Green innovation in multiproduct firms, environmental policy, and sustainable investing

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Abstract

This paper addresses the disconnect between the macroeconomic literature on the green transition and the finance literature on sustainable investing by funds and individual investors, as well as green-asset purchases by central banks. It tests the hypothesis that raising the cost of capital for multiproduct firms engaged in both dirty and clean production may hinder green innovation and slow the green transition. It develops a parsimonious perfect-foresight general-equilibrium model with directed technical change to evaluate the effects of the change in the cost of capital on emissions and innovation. In the context of multiproduct firms involved in both dirty and clean production and innovation, I show that while a rise in dirty input taxes and a rise in a clean research subsidy reduce emissions, an increase in the cost of capital yields an ambiguous outcome: it amplifies the existing asymmetry in clean versus dirty research productivity. The direction of change from this policy depends on the initial relationship between the productivities of clean and dirty innovation. Thus, if dirty innovation is more productive than clean innovation, a higher cost of capital makes firms focus more on dirty innovation, potentially disrupting the green transition. Thus, central bank green-asset purchases and ESG investing strategies should be based not only on a company's ESG score or emissions intensity, but also on the degree of substitution between clean and dirty production, sectoral innovation advancement, and the structure of the innovation process.

Keywords: green innovation, directed technical change, ESG investing, central bank, green quantitative easing, optimal environmental policy.

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1 Introduction

A growing literature examines factors that facilitate the "green" transition, mostly focusing on optimal carbon taxes, research subsidies, tax credits, and impact investing, aimed at reducing the cost of capital for "green" firms. Despite advances in both general equilibrium models (featuring real decisions and the innovation process of the firms) and the finance literature (examining the effects of ESG investing and green quantitative easing on returns), there remains a disconnect between these two bodies of work. Specifically, while the former generally focuses on optimal taxation without a detailed treatment of the financial sector, the latter typically studies asset pricing and the cost of capital without accounting for the decisions of "dirty" firms to pollute and their potential for transitioning to "green" technologies. As such, there remains a need for research that bridges these gaps and provides a more comprehensive understanding of the interplay between technological change, financial markets, and environmental policy.

Optimal government policy is usually implemented through carbon taxation, innovation subsidies, and tax credits for installing green technology, which show the expected result in both theoretical models (Golosov et al., 2014; Acemoglu et al., 2012) and the data. At the same time, uncertainty remains about the effects of increasingly popular impact investing with an ESG (Environmental, Social, Governance) agenda and green asset purchases by central banks. The prevailing view is to rank companies by their ESG score and divest from "dirty" companies by increasing their cost of capital, while investing more in "clean" companies to lower their cost of capital. Although this approach is theoretically reasonable, it has two main drawbacks. First, the empirical paper by Hartzmark and Shue (2022) shows that when the cost of capital rises for brown firms, pollution increases. Furthermore, Cohen et al. (2021) provide evidence that dirty firms, primarily those in the energy sector, are capable of generating high-quality green innovations. They find a positive correlation between a company's greenhouse gas emissions and its green patenting activity. Notably, ESG investors appear to "reward" (resulting in a lower cost of capital) for innovation only the green firms. Taken together, these two pieces of evidence suggest testing the *hypothesis* that increasing the cost of capital for firms with both "dirty" and "clean" production can cause them to pollute more and innovate less.

Second, it is unclear what policy regarding changing the cost of capital to implement when a company is neither conventionally clean nor dirty: it involves both types of production and innovation (I will call them "mixed" companies). Increasing the cost of capital for such a company may also suppress green innovation and slow the green transition. A suitable example is the car industry. For instance, Toyota has the highest number of green patents

in the car industry but has higher emission intensity than Tesla. Thus, increasing the cost of capital for such a company slows the green transition. Additionally, by analyzing green patents from the Orbis Intellectual Property dataset and S&P ESG scores for car companies, I found an ambiguous relationship between green patents and car companies' ESG scores: there is no clear positive correlation between them in Figure 1. Therefore, by increasing the cost of capital for companies with low ESG scores, we can reduce the green innovation of companies that previously patented heavily.

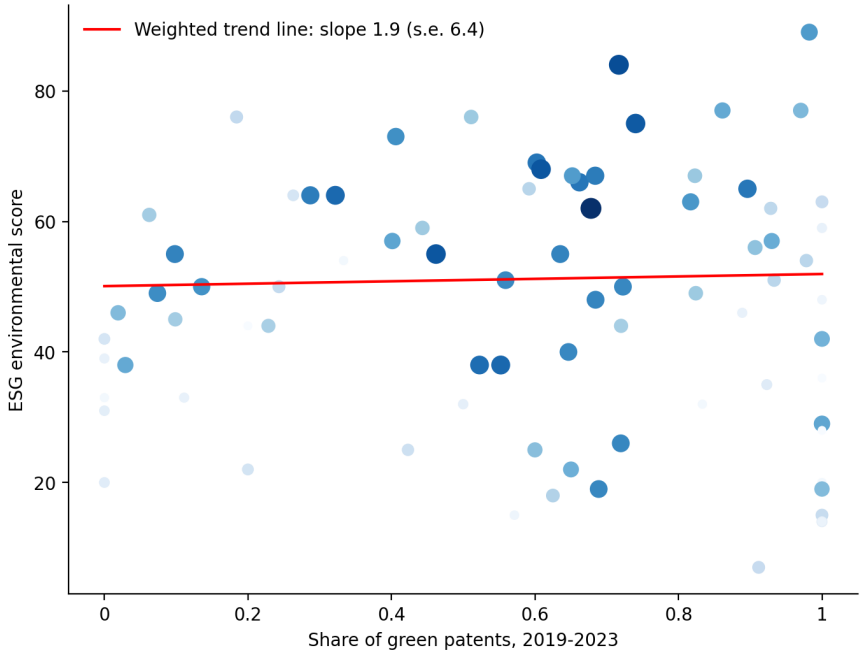


Figure 1. The ambiguous relationship between green patents and ESG scores of car companies Computed from Orbis Intellectual Property (main IPC code of each patent publication, 2019–2023) and S&P Global ESG environmental scores (2022) for car companies. Dot size and regression weights are the log of the number of green and brown patents; green and brown patents are defined in Appendix A.2.

There are different reasons why brown firms issue so many green patents. First, it can indicate a true green transition or an effect of the government's environmental policy. There can also be spillovers between clean and dirty innovations (knowing how to produce an internal combustion engine car will help to produce an electric vehicle, as it is also a car), which affects the innovation frontier of the company and is one of the main focuses of the paper. There can also be a preemptive invention or "sleeping patents", different patent quality, but this did not find much support in the literature (Cohen et al., 2021). Furthermore, there can be costly signaling to attract ESG investors. Thus, for a low level of government regulation, divestment is the only reasonable strategy because of an adverse selection problem, while it makes green innovation harder. There are also various opinions regarding using patenting activity as a metric for green innovation, but it is still better than

other data, such as the firm's general R&D expenses, and for certain industries, such as the car industry, it is the most suitable metric.

In this paper, I want to answer the following **research questions.** *How will the change in the cost of capital affect the emissions of "mixed" firms over time? Specifically, is there a "green paradox" of conventional ESG investing and green asset purchases by central banks: can these measures negatively affect innovation in production? Can directed technical change and spillovers between dirty and green sectors explain brown firms' green transition? How does government regulation interplay with ESG investing?*

To answer these questions, I build a parsimonious general equilibrium model with directed technical change in clean and dirty productivities with the following features. The economy has three sectors: a unit mass of monopolistically competitive firms producing varieties, consumers with CES preferences over varieties, and the government. The output of the firm is a mix of two types of production: dirty (for example, making fuel cars), which increases carbon emissions, and clean (for example, making electric vehicles), and they are aggregated via the CES technology. Additionally, for each type of production, there is a nested Cobb-Douglas technology of machines used to produce an output and their productivity. The defining feature of this model, in contrast to the existing literature, is the fact that I do not consider separately clean and dirty firms, but rather "mixed" firms, which produce both clean and dirty output. It will allow me to understand the effect of ESG investing and, in the future, green asset purchases by central banks on the whole company (as we cannot separately affect the cost of capital for clean and dirty production of the firm by standard investing strategies). Innovation in the productivity of machines features the directed technical change technology frontier with semi-endogenous growth. In other general equilibrium models, there is either strong path dependency of innovation, meaning that the technology level in one sector is unaffected by the technology level in another sector, or strong scale effects (increase in the number of researchers is increasing growth rates) of the innovation process, which is shown to be not realistic in the data (Jones, 1995; Bloom et al., 2020). Therefore, I model an innovation process without scale effects and allow for cross-spillovers between clean and dirty technologies.

One of the paper's main results is a parameter condition for the stability or instability of the resulting optimal path and decentralized equilibrium. I find that the higher the degree of substitution between clean and dirty production in the firm's total output and the lower the cross-sector spillover in the innovation process, the higher the chance of a corner solution in the innovation process, where only one type of innovation and production is implemented. In contrast to the existing literature, I find that with any positive cross-sector spillover in the innovation sector, research never concentrates entirely in one sector; in the calibration,

the BGP with both types of production is unique and stable. Thus, we need to estimate the innovation process of mixed firms structurally, as it has important implications for optimal policy.

To facilitate the green transition in the decentralized equilibrium of my model, the government might opt to raise the carbon tax rate for dirty inputs, provide a clean research subsidy, offer a revenue subsidy to compensate for the monopoly distortion, or directly change the cost of capital (discount rate) for firms by offering subsidized loans or implementing stricter financial regulations. Any tax revenue generated from these policies is then redistributed back to consumers. Consumers, unlike the social planner, do not fully account for the emissions externality, which is why taxation is needed.

I find that, consistent with the literature, a rise in the dirty input tax and a clean research tax subsidy reduce emissions, and, importantly, an increase in the cost of capital amplifies the existing asymmetry in clean versus dirty research productivity. In contrast to the dirty input tax and a clean research subsidy, the ESG investing strategy's results depend on the relationship between innovation productivities. If there is more advancement in dirty innovation, an increase in the cost of capital makes the ratio of clean to dirty productivities lower; thus, firms focus more on dirty innovation. Thus, raising the cost of capital for firms with initial dirty-technology advancement can lead to a "Green Paradox," where higher capital costs increase emissions and disrupt the green transition. In the current calibration, this effect is small relative to the effect of a dirty input tax.

Therefore, I contribute to the existing literature by evaluating the effect of changes in the cost of capital of mixed firms in a general equilibrium model with a detailed innovation frontier. In future steps, I will structurally estimate the innovation process and enrich the model with more realistic ESG preferences of investors to match the observed effect on the cost of capital in the data.

Related Literature. The paper is aligned with the vast literature on climate externalities and optimal taxation. In 1994, Nordhaus developed the DICE model (Nordhaus, 1994), a pioneering study that integrates a dynamic model of climate change and the economy by expanding the neoclassical Ramsey model to include emissions and climate change equations. Hassler et al. (2022), Golosov et al. (2014), and Krusell and Smith (2022) developed more advanced general equilibrium frameworks to evaluate the optimal Pigouvian tax and model dirty and clean energy firms more precisely, but they do not include a financial sector to assess the impact of ESG investors.

Acemoglu et al. (2012) expanded this framework by introducing endogenous technical change and included two intermediate sectors: "dirty" and "clean". In such a two-sector

model, market forces would naturally favor the sector that is already more advanced. As a result, the social planner needs to redirect innovation from the dirty to the clean sector to reduce emissions in the long run. Their research reveals that sustainable growth can be attained with temporary taxes/subsidies that steer innovation toward clean inputs when inputs are highly substitutable. The optimal policy combines carbon taxes and research subsidies to prevent excessive reliance on carbon taxes. Acemoglu (2023) added spillovers between clean and dirty technologies, while the innovation process still has scale effects, which have been shown to be unrealistic (Bloom et al., 2020). Aghion et al. (2016) built a static model to show the effect of taxation on the car industry and estimate the innovation frontier for this industry using patent data. *I contribute to the general equilibrium literature* by adding an endogenous growth component, directed technical change with cross-sector spillovers; and also *to the climate endogenous growth literature* by adding multiproduct firms rather than strictly clean or dirty and evaluating the effect of the change in cost-of-capital on innovation and production.

Relevant finance literature mainly focuses on the effect of impact investment on the cost of capital. Berk and van Binsbergen (2021) establish a parsimonious equation for changes in the cost of capital that depends on three inputs: the proportion of socially responsible capital, the percentage of targeted firms in the economy, and the correlation between targeted firms and the overall stock market. Based on the current state of ESG investment, their findings suggest that the effect on capital costs is too small to meaningfully affect investment decisions.

Pedersen et al. (2021) have developed a partial-equilibrium model that incorporates investors whose preferences are influenced by ESG scores. Similarly, Pástor et al. (2021) developed a model with agents who have ESG preferences and examined the security market line within this framework. Their findings suggest that agents with stronger ESG preferences, favoring green assets over brown, experience lower expected returns. However, their study does not address whether the actions of ESG investors effectively achieve socially optimal outcomes. I plan to incorporate the micro-foundation of ESG preferences into my model.

Two other papers address the concerns regarding the optimality of the current sustainable investing strategies. Cohen et al. (2021) present findings that challenge prevailing notions by revealing that oil, gas, and energy-producing firms, often excluded from ESG funds due to lower scores, are pivotal innovators in the U.S. green patent landscape. These firms, despite lower ESG ratings, exhibit substantial and high-quality contributions to green innovation, positioning them as influential first-movers in critical areas like carbon capture. Hartzmark and Shue (2022) empirically demonstrate that reducing financing costs for already green firms yields modest improvements in impact at best. Conversely, increasing the financing costs

for brown firms results in substantial negative changes. This suggests that diverting capital from brown to green firms in sustainable investing may be counterproductive, intensifying the brown nature of the firms excluded from the portfolio without significantly enhancing the green attributes of the included ones.

Other relevant papers address the implication of allowing "dirty" firms to innovate and become greener. Heinkel et al. (2001) developed a model in which ESG investors' investment behavior raises the cost of capital for polluting firms and lowers it for green firms. Firms then opt to become green at a cost that is determined endogenously. Barnea et al. (2005) also considered firm investment behavior and analyzed how divestiture affects total investment.

To this end, I intend to examine technological innovation in renewable energy production and empirically investigate how a lower cost of capital affects innovation and, in turn, improves the efficiency of renewable energy production. For instance, Xu (2020) present evidence that higher economic policy uncertainty hinders innovation not only through the conventional investment irreversibility channel but also through the cost-of-capital channel. In other words, government economic policy uncertainty increases firms' cost of capital, which reduces innovation.

My paper *contributes to the finance literature* by analyzing the cost-of-capital channel and innovation in a general equilibrium model. As a future step, I am planning to assess the costly signaling channel of green innovation (to attract ESG investors), thus I will explore further the literature on adverse selection and greenwashing, where the relevant frameworks are Biais and Landier (2022), Cicala et al. (2022), Oehmke and Opp (2025).

The rest of the paper is organized as follows. Section 2 presents a basic framework of directed technological change in mixed firms, and characterizes the properties of the social planner's allocation and optimal innovation. Section 3 presents the model's decentralized equilibrium, the main results, and how they differ from the social planner's solution. Section 4 outlines the next steps, Section 5 concludes, and the appendix presents proofs, additional derivations, and results.

2 Model

The infinite-horizon continuous-time economy is composed of a unit mass of symmetric firms that each produce a variety demanded by a representative consumer that has a CES aggregator over varieties. This setup will allow us to model monopolistic competition in decentralized equilibrium. Each variety can be represented as a product of a firm. A suitable example is the car industry, where the purchased variety is a Toyota car, a Tesla car, etc.,

and a firm can produce both fuel cars and electric vehicles. Thus, the output of the companies' production is a mix of "dirty" production (internal combustion engine cars), which makes high emissions, and "clean" production (electric vehicles), which pollutes less. Such a setup is more realistic than having strictly clean and dirty companies, as in the standard environmental macroeconomic models. It allows us to study such policies as changing the cost of capital for the whole company when it is almost impossible to change the cost of capital only for clean or dirty investments of the company by standard investment strategies.

Specifically, an output y_{it} of the firm i is produced using the clean and dirty machines, x_{cit} and x_{dit} respectively, and productivity for each type of machine, A_{cit} and A_{dit}, with their constant shares of β and $1-\beta$. Clean and dirty production is aggregated via the CES production function, and $\varepsilon > 0$ measures the degree of substitution of clean and dirty parts of the firm's production:

$$y_{it} = \left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{1}$$

The consumer maximizes a discounted flow of utility from aggregate consumption per capita C_t/L_t in each period, and receives a disutility from emissions S_t due to the dirty part of the firms' production:

$$\int_0^\infty e^{-(\rho-n)t} [u(C_t/L_t) - v(S_t)] dt$$

Here $C_t = \left(\int_0^1 C_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}$ is consumption of the unit mass of varieties, $\sigma > 1$ is the degree of substitution of varieties. n is the growth rate of population L_t, ρ is the discount rate. We assume logarithmic utility from consumption and logarithmic disutility of emissions (to derive an exact balanced growth path of the economy), where the latter is amplified by parameter θ so that the preferences become:

$$\int_0^\infty e^{-(\rho-n)t} [log(C_t/L_t) - \theta log(S_t)] dt \tag{2}$$

The stock of emissions is increased from the dirty production of the firm, and a part δ of carbon emissions is naturally or industrially captured each period. Thus, the stock of emissions evolves as:

$$\dot{S}_t = \int_0^1 (1-\delta) A_{dit}^{(1-\beta)} x_{dit}^{\beta} di \tag{3}$$

Detailed characterization of the carbon emissions in line with Golosov et al. (2014), Nordhaus

(1994), and Acemoglu et al. (2016) is presented in Appendix A.1.

Returning to production, using both clean and dirty machines requires ψ units of output; thus, output is split between machine consumption and production. Therefore, the resource constraint (market clearing for the total output) becomes:

$$\int_0^1 C_{it}di + \psi \int_0^1 x_{cit}di + \psi \int_0^1 x_{dit}di = \int_0^1 y_{it}di \tag{4}$$

The productivity of machines is governed by an innovation possibilities frontier similar to that in the directed technical change literature. The increase in productivity A_{jit} in sector $j \in \{c, d\}$ comes from the probability of innovation happening (or, alternatively, micro-founded as an innovation spillover, or productivity of innovation unconditional of resources), η_j, from the stock of existing innovations (patents) A_{jit} and the stock of existing innovations in another sector of production A_{-jit} and of the number of researchers z_{jit} working in this sector. Parameters α and γ measure the strength of the own- and cross-sector innovation spillover.

$$\dot{A}_{jit} = \eta_j z_{jit} A_{jit}^{\alpha} A_{-jit}^{\gamma} \tag{5}$$

The household supplies labor L_t inelastically, which is allocated between researchers in clean and dirty innovation. Market clearing for researchers is as follows:

$$\int_0^1 z_{cit}di + \int_0^1 z_{dit}di = L_t, \ \frac{\dot{L}}{L} = n \tag{6}$$

Importantly, the innovation process here has no scale effects. The scale-effect property, present in other papers in the literature, means that increasing the *number* of researchers increases the *growth* rate of innovations. My model of the innovation process without scale effects is similar to Jones (1995), but I add a cross-sector spillover and retain the property that "ideas are harder to find". Here, this is expressed by setting the own-innovation spillover parameter α to be less than 1; thus, the larger the stock of existing innovation, the harder it is to produce a new innovation. The cross-sector spillover parameter γ captures the complementarity of innovations between the two sectors. For a balanced growth path to exist, we need $\alpha + \gamma < 1$, assuming positive population growth n. The most closely related innovation process from the literature is Acemoglu (2023), where the innovation process would be analogous to $\dot{A}_{jit} = \eta_j z_{jit} A_{jit}^{(1+\delta_A)/2} A_{-jit}^{(1-\delta_A)/2}$ for $j \in \{c, d\}$, where $\alpha \equiv (1 + \delta_A)/2$, $\gamma \equiv (1 - \delta_A)/2$ to equate with our setting. This features scale effects, which are not in line with the data (Jones, 1995; Bloom et al., 2020).

The full economic environment is presented in the following table:

Firm production	$y_{it} = \left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}$
Utility	$\int_0^\infty e^{-(\rho-n)t}[u(C_t/L_t) - v(S_t)]dt$
Consumption per capita	$c_t = \left(\int_0^1 c_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}, \ \sigma > 1$
Flow utility	$u(c_t) = log(c_t)$
Carbon cycle	$\dot{S}_t = \int_0^1 (1-\delta) A_{dit}^{(1-\beta)} x_{dit}^{\beta} di$
Disutility from emissions	$v(S_t) = \theta log(S_t)$
Resource constraint	$\int_0^1 C_{it} di + \psi \int_0^1 x_{cit} di + \psi \int_0^1 x_{dit} di = \int_0^1 y_{it} di$
Innovation	$\dot{A}_{cit} = \eta_c z_{cit} A_{cit}^{\alpha} A_{dit}^{\gamma}$, $\dot{A}_{dit} = \eta_d z_{dit} A_{dit}^{\alpha} A_{cit}^{\gamma}$
Labor resource constraint	$\int_0^1 z_{cit} di + \int_0^1 z_{dit} di = L_t, \quad \frac{\dot{L}}{L} = n$

Table 1. Economic environment

2.1 Social planner's problem

In this subsection, I characterize the social planner's solution, where, in contrast to the decentralized equilibrium (described further below), the social planner internalizes emissions from the dirty part of production and can redirect machines and innovation from one sector to another.

Definition. *The socially optimal allocation is a dynamic path of variety production y_{it}, consumption C_t, machines x_{jit}, researchers allocations z_{jit}, carbon emissions S_t, and productivities of machines A_{jit} that maximizes the intertemporal utility of the representative consumer, (2), subject to (1), (3), (4), (5), (6) and transversality conditions.*

The full social planner's problem looks as follows:

$$\max_{\{C_{it},x_{cit},x_{dit},z_{cit},z_{dit},S_t\}_0^\infty} \int_0^\infty e^{-(\rho-n)t}\left[log\left[\left(\int_0^1 C_{it}^{\frac{\sigma-1}{\sigma}}di\right)^{\frac{\sigma}{\sigma-1}}\right]-log(L_t)-\theta log(S_t)\right]dt$$

s.t.

$$\int_0^1 C_{it}di+\psi\int_0^1 x_{cit}di+\psi\int_0^1 x_{dit}di=$$

$$=\int_0^1\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}+(A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}di$$

$$\dot{A}_{cit}=\eta_c z_{cit}A_{cit}^{\alpha}A_{dit}^{\gamma},$$

$$\dot{A}_{dit}=\eta_d z_{dit}A_{dit}^{\alpha}A_{cit}^{\gamma}$$

$$\int_0^1 z_{cit}di+\int_0^1 z_{dit}di=L_t$$

$$\dot{S}_t=\int_0^1(1-\delta)A_{dit}^{(1-\beta)}x_{dit}^{\beta}di$$

$$TVC:$$

$$\lim_{t\to\infty}e^{-(\rho-n)t}\mu_{1t}A_{cit}=0,\qquad \lim_{t\to\infty}e^{-(\rho-n)t}\mu_{2t}A_{dit}=0,$$

$$\lim_{t\to\infty}e^{-(\rho-n)t}\xi_t S_t=0,$$

Properties of the Social planner's solution:

The social planner's solution in such a setting is characterized by a Balanced Growth Path (BGP), unique under the conditions of Proposition A1 (which hold in the calibration), where the optimal ratio of productivities (cutoff) $\left(\frac{A_{cit}}{A_{dit}}\right)^*$ and the optimal ratio of machines used in clean and dirty sectors $\left(\frac{x_{cit}}{x_{dit}}\right)^*$ are constant over time. The BGP ratio represents a condition for innovation in both sectors with the same innovation growth rate.

Importantly, even when the BGP ratio is unique, the optimal allocation can be stable or unstable depending on the degree of substitution between clean and dirty parts of production and the own-cross spillover parameters in the innovation process.

The following proposition is one of the main results of the paper:

Proposition. *Consider the planner without the emission externality, which has the same research allocation as the decentralized equilibrium without taxes (Lemma 2 in Appendix A.6.1). Let* $\varepsilon^* = 1+\frac{1-(\alpha-\gamma)}{(\alpha-\gamma)(1-\beta)}$. *(i) If* $\gamma=0$, *the BGP is unique. Suppose only the ratio of technologies starts off the BGP (Proposition A3 states the exact condition). If* $\varepsilon<\varepsilon^*$, *the economy converges to the BGP: all researchers work in the lagging sector until the BGP ratio is reached, and from then on research is split as on the BGP. If* $\varepsilon>\varepsilon^*$, *the BGP is*

unstable, and if initially $\left(\frac{A_{ci0}}{A_{di0}}\right) > \left(\frac{A_{cit}}{A_{dit}}\right)^*$ *the path toward the clean corner,* $\left(\frac{A_{cit}}{A_{dit}}\right) \underset{t\to\infty}{\to} \infty$*, dominates every path that stays above the BGP, and vice versa. (ii) For any* γ *and* $\eta_c = \eta_d$*, staying on the symmetric BGP satisfies the necessary second-order condition for optimality if* $\alpha - \gamma - \frac{1}{(1-\beta)\varepsilon+\beta} < 2\gamma(1+\gamma-\alpha)\frac{g_A}{\rho}$*, and violates it, so that the BGP is unstable, if the inequality is reversed. (iii) If* $\gamma > 0$*, research never concentrates in one sector; the BGP is unique and stable for every* ε *in the basic calibration, but when the condition in (ii) fails, the economy settles on an interior BGP where one technology dominates. (iv) If* $\gamma < 0$*, corner paths satisfy the first-order conditions for any* $\varepsilon > 1$*, and by (ii) the mixed BGP is unstable already for some* $\varepsilon < \varepsilon^*$*.*

Appendix A.6 proves (i) in Proposition A3 and (ii) in Proposition A2, and gives conditions for existence and uniqueness of the BGP in Proposition A1. The first-order conditions are not sufficient here, so I also solve the planner's problem numerically on a grid; this global solution confirms (ii)–(iv) in eight parameter cases, and in three cases with emissions.

The Proposition also applies to the decentralized equilibrium without taxes; with constant taxes, the BGP and corner conditions keep the same form (Proposition A4). The Proposition means that the higher the degree of substitution between two sectors ε, the less the firm needs both sectors and the easier it is to give up innovation in one sector for more efficient production. The cross-sector spillover γ works in the opposite direction: with $\gamma > 0$ research in each sector raises the productivity of research in the other, so the planner never abandons either sector; with $\gamma < 0$ the stock of one technology hampers research in the other, so corner paths satisfy the first-order conditions as soon as the two outputs are substitutes ($\varepsilon > 1$).

An illustration of the stability of the optimal allocation and further policy implications is presented in Figure 2. For a stable optimal path, if we start with the ratio of productivities $\left(\frac{A_{ci0}}{A_{di0}}\right)$ below the cutoff $\left(\frac{A_{cit}}{A_{dit}}\right)^*$, the economy will converge to the optimal cutoff over time. Thus, going to the optimal policy question, if the optimal policy wants to target lower carbon emissions (which are increased each period by $A_{dit}^{(1-\beta)}x_{dit}^{\beta}$), the ratio of productivities of clean to dirty sectors should be higher, and the economy will converge to this higher ratio, making emissions lower.

Now consider an unstable optimal path. We see in the upper-right picture that if we start with the ratio of productivities $\left(\frac{A_{ci0}}{A_{di0}}\right)$ below the cutoff $\left(\frac{A_{cit}}{A_{dit}}\right)^*$, the economy will converge to the left corner with innovation directed only into dirty technology. Thus, starting anywhere below the cutoff leads to the dirty corner, and emissions per period increase over time. Therefore, the optimal policy should shift the cutoff $\left(\frac{A_{cit}}{A_{dit}}\right)^*$ to the left, below the initial ratio, for the economy to converge to the clean right corner, where there will only be innovation directed to increasing the productivity of clean production. The initial ratio that

separates the two corners is the ratio at which the planner is indifferent between them; it equals the cutoff in symmetric cases and can differ otherwise (Appendix A.6.6).

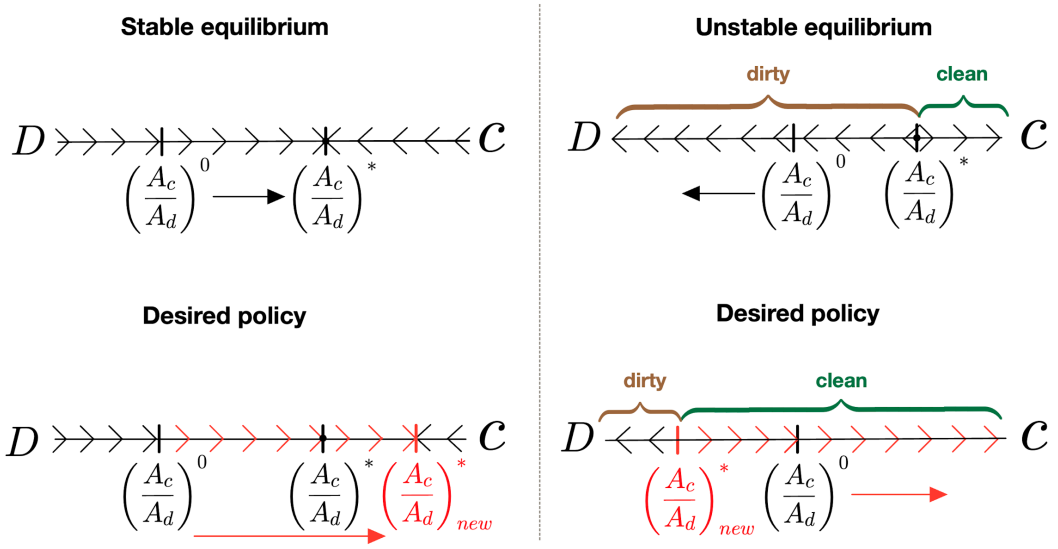


Figure 2. Illustration of stable and unstable equilibria and required optimal policies to reduce emissions

An additional important observation here is that regardless of the elasticity of substitution of clean and dirty parts of production, if there is a positive cross-sector spillover, meaning that $\gamma > 0$, we will need both types of innovation to happen; thus, the firms will produce both "dirty" and "clean" cars in the car industry example (with large ε the economy can still settle on a BGP where one type dominates). This contrasts with models where the innovation process had scale effects (less realistic in the data). For instance, in my model, for a corner optimal path I need $\gamma \equiv (1-\delta_A)/2 \leq 0 \Rightarrow \delta_A \geq 1$, while the assumption in Acemoglu (2023) is that $\delta_A \leq 1$, and for instability in his model it has to be that $\sigma_A > 1/\delta_A$, where σ_A is the elasticity of substitution between clean and dirty inputs in his paper. This means that, for Acemoglu (2023), even with positive cross-spillover, an unstable (corner) optimal path can exist, whereas without scale effects in the innovation process, this effect disappears. For the innovation process without scale effects, even a small positive cross-sector spillover ($\gamma > 0$) will make this sector's progress necessary for innovation, thus ruling out allocations where only one type of innovation is performed. This means it is crucial to further investigate the innovation process in the data for different sectors, as the functional form of the innovation process strongly affects the optimal allocations and policy.

While a detailed proof of the existence of corner solutions is presented in the appendix, the general idea is as follows.

Consider the following Hamiltonian for the social planner's problem, where the optimal allocation is symmetric (all the firms are the same).

$$\begin{aligned} H =& log(C_t/L_t) - \theta log(S_t) + ... \\ &+\mu_{1t}[\eta_c z_{cit} A_{cit}^{\alpha} A_{dit}^{\gamma}] + \mu_{2t}[\eta_d [L_t - z_{cit}] A_{dit}^{\alpha} A_{cit}^{\gamma}] \end{aligned}$$

where z_{cit} are researchers making green innovations, and $L_t - z_{cit}$ are researchers engaged in dirty innovation. For the corner solution, we need all researchers to be employed only in one sector ($z_{jit} = L_t, z_{-jit} = 0$). After solving the Hamiltonian, we derive the parameter conditions for this corner solution to exist.

The key first-order condition for the dynamic problem for determining the corner solution is the one with respect to z_{cit}. We have three possibilities: a clean corner, an interior solution, or a dirty corner.

$$H_{z_{cit}} : \mu_{1t}[\eta_c A_{cit}^{\alpha} A_{dit}^{\gamma}] - \mu_{2t}[\eta_d A_{dit}^{\alpha} A_{cit}^{\gamma}]$$

$$\begin{cases} > 0 \;\; \Rightarrow & z_{cit} = L_t & \Rightarrow \;\; \frac{A_{cit}}{A_{dit}} \to \infty \; (corner) \\ = 0 \;\; \Rightarrow & z_{cit} \in (0, L_t) & \Rightarrow \;\; \frac{A_{cit}}{A_{dit}} = \left(\frac{A_{cit}}{A_{dit}}\right)^* \; (interior) \\ < 0 \;\; \Rightarrow & z_{cit} = 0 & \Rightarrow \;\; \frac{A_{cit}}{A_{dit}} \to 0 \; (corner) \end{cases}$$

Let's consider the condition for the economy to converge to a clean corner. Then $\mu_{1t}[\eta_c A_{cit}^{\alpha} A_{dit}^{\gamma}] - \mu_{2t}[\eta_d A_{dit}^{\alpha} A_{cit}^{\gamma}]$ has to be greater than 0. After substituting the remaining first-order conditions, we get the following inequality[1]:

$$\frac{\left(\frac{A_{cit}}{A_{dit}}\right)^{\alpha-\gamma-\frac{1}{(1-\beta)\varepsilon+\beta}}}{\rho - n - \frac{\dot{\mu}_{1t}}{\mu_{1t}} - \alpha g_{A_c}} \frac{\eta_c}{\eta_d} > \frac{1}{\rho - n - \frac{\dot{\mu}_{2t}}{\mu_{2t}}} + \frac{\left(\frac{A_{cit}}{A_{dit}}\right)^{\frac{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}{1-\beta\frac{\varepsilon-1}{\varepsilon}}}}{\left(\rho - n - \frac{\dot{\mu}_{1t}}{\mu_{1t}} - \alpha g_{A_c}\right)\left(\rho - n - \frac{\dot{\mu}_{2t}}{\mu_{2t}}\right)} \gamma g_{A_c}$$

We also know that when $z_{cit} = L_t, z_{dit} = 0, \;\; g_{A_c} > g_{A_d}$, thus $\frac{A_{cit}}{A_{dit}} \to \infty$.

The condition for the elasticity of substitution between dirty and clean production to have the unstable optimal path ε^* is derived from the restriction on the parameters for the

[1]Here I consider the social planner's problem without internalizing the emissions externality for tractability in the main text. Appendix A.3.2 derives the dirty corner with emissions, which has the same restriction on parameters; Appendix A.6.6 shows that with emissions the clean corner satisfies the first-order conditions in more cases, but in the case solved globally ($\gamma = 0$, $\varepsilon = 2.5$) the economy still converges to the mixed BGP.

inequality above to hold:

$$\begin{aligned} &or \;\; \alpha - \gamma - \frac{1}{(1-\beta)\varepsilon + \beta} > 0, \;\; and \;\; \gamma \leq 0; \\ &or \;\; \alpha - \gamma - \frac{1}{(1-\beta)\varepsilon + \beta} \leq 0, \;\; and \;\; \gamma < 0, \;\; and \;\; \frac{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}{1-\beta\frac{\varepsilon-1}{\varepsilon}} > 0 \end{aligned}$$

Simplifying, we get:

$$\begin{aligned} &or \;\; \varepsilon > \frac{1-\beta(\alpha-\gamma)}{(1-\beta)(\alpha-\gamma)} = 1 + \frac{1-(\alpha-\gamma)}{(1-\beta)(\alpha-\gamma)}, \;\; and \;\; \gamma \leq 0 \\ &or \;\; \varepsilon \leq 1 + \frac{1-(\alpha-\gamma)}{(1-\beta)(\alpha-\gamma)}, \;\; and \;\; \gamma < 0, \;\; and \;\; \varepsilon > 1 \end{aligned}$$

The result is symmetric for the dirty corner solution.

2.1.1 Results for the social planner's problem

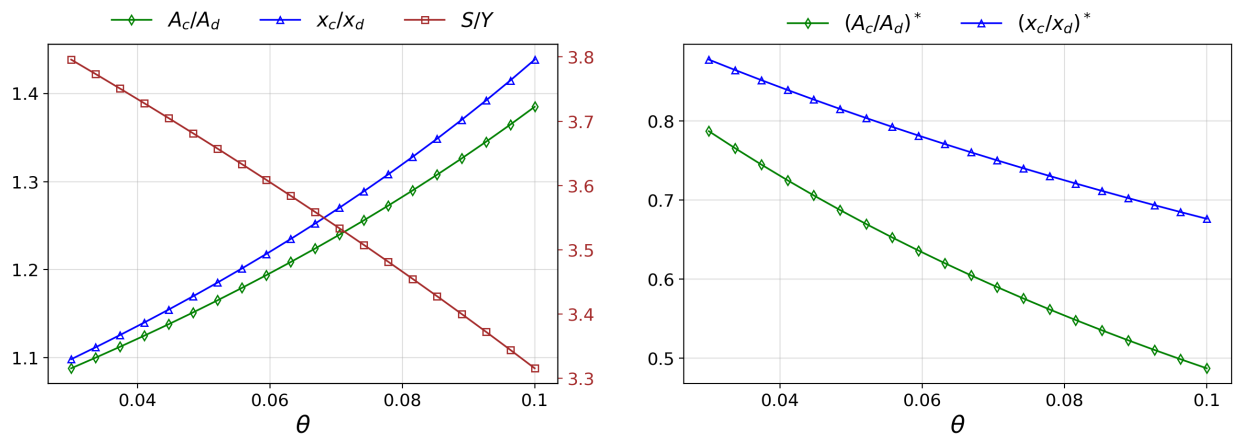
For the basic calibration below, I run an experiment varying the disutility-of-emissions parameter θ (the higher θ is, the higher the disutility of emissions) to see whether the model works as expected.

Basic calibration:

Parameter	α	γ	β	ρ	n	δ	η_c	η_d	ψ	ε	σ	θ	ε^*
Value	1/3	1/6	1/3	0.02	0.01	0.8	0.1	0.1	1/3	4	4	0.03	8.5

With $\gamma = 1/6 > 0$, this calibration lies in the stable region for any ε (part (iii) of the Proposition), so $\varepsilon^* = 8.5$ in the table is not a stability threshold here. The unstable case changes only the cross-sector spillover, $\gamma = -1/15$, so that $\gamma < 0$ and $\varepsilon = 4 > \varepsilon^* = 3.25$. In Section 3, Figure 4 uses $\eta_c = 0.105$, $\eta_d = 0.1$, and Figures 5–7 use $\eta_c = 0.11, \eta_d = 0.1$ (left panels) and $\eta_c = 0.1, \eta_d = 0.11$ (right panels).

We can see in Figure 3 that for the case where the production function and innovation frontier parameters ensure a stable optimal allocation and the disutility of emissions θ increases from 0.03 to 0.1, the ratio of technologies increases from 1.09 to 1.38; thus, emissions per output decrease by 13%. This means that the model's predictions hold. In the unstable optimal allocation, increasing θ lowers the cutoff ratio of technologies from 0.79 to 0.49; thus, the economy converges more easily to the clean corner. The emissions plot is irrelevant here: if the ratio of technologies becomes small enough, below the initial ratio, the economy converges to the clean corner, where emissions no longer grow.



Stable optimal allocation

Unstable optimal allocation ($\gamma = -1/15$)

Figure 3. Comparative statics for an increase in the disutility of emissions on the optimal allocation S/Y is the balanced-growth-path stock of emissions over output. Basic calibration with $\eta_c = \eta_d = 0.1$; the right panel uses $\gamma = -1/15$.

3 Decentralized equilibrium

This section presents the decentralized equilibrium of the model.

Agents:

Consumers maximize the discounted flow of utility from consumption per capita c_t and disutility from emissions S_t. The difference from the social planner's solution is that agents face the disutility of emissions, but cannot affect emissions and do not internalize them.

The evolution of the household's assets is as follows. The agent receives income from supplying labor hours L_t inelastically, doing research to make innovations and earning wage w_t, spending on varieties c_{it} priced at p_{it}, and receiving lump-sum net transfers T_t from the government. The accumulated savings earn interest at rate r_t, and the household's assets are invested in the companies it holds. The agent's problem looks as follows:

$$\max_{c_{it}} \int_0^\infty e^{-(\rho-n)t} \left[log(c_t) - \theta log(S_t)\right] dt, \qquad c_t = \left(\int_0^1 c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}$$
$$s.t.\ \dot{a}_t = ra_t - L_t \int_0^1 p_{it} c_{it} di + w_t L_t + T_t, \quad a_t = \int_0^1 V_{it} di$$

Firms: A monopolistically competitive firm producing variety i maximizes its value, the discounted flow of profits. The firm faces an inverse demand for the variety y_{it} from the consumer and sets the output price p_{it}, thus receiving the revenue $p_{it}y_{it}$. Each period, the firm faces expenses from using the machines and hiring researchers to produce innovation.

In such a decentralized equilibrium, the government has four policy instruments:

- $\tau_{cost,t}$ – affects the interest rate by which the firm discounts profits (cost of capital of the firm). $\tau_{cost,t} > 1$ increases the firm's cost of capital, making investments more expensive and present profits more valuable.

- $\tau_{rev,t}$ – subsidy/tax on revenue. Such an instrument can correct the monopoly distortion. Because of monopoly power, the firm will choose lower machine use and thus produce less output than in the social planner's solution to extract the highest profits. Therefore, by subsidizing the revenues ($\tau_{rev,t} > 1$) we can achieve higher use of machines, output, and consumption. The drawback of such a policy is that it increases the use of all machines, so total emissions will also increase.

- $\tau_{br,t}$ – carbon tax/subsidy imposed on the use of dirty machines. $\tau_{br,t} > 1$ means that it is costlier to use dirty machines and produce a dirty part of the output.

- $\tau_{cl,t}$ – tax/subsidy for hiring researchers engaged in clean innovation. $\tau_{cl,t} < 1$ means that it is less costly to hire researchers engaged in clean innovation.

The firm's problem looks as follows:

$$\max_{x,z,p} V_{i0} = \int_0^\infty e^{-r_t \overbrace{\tau_{cost,t}}^{\text{cost of capital}}} \pi_{it} dt =$$

$$= \int_0^\infty e^{-r_t \tau_{cost,t} t} [\overbrace{\tau_{rev,t}}^{\text{revenue subsidy}} p_{it} y_{it} - \psi(x_{cit} + \overbrace{\tau_{br,t}}^{\text{carbon tax}} x_{dit}) - w_t(z_{cit} \overbrace{\tau_{cl,t}}^{\text{clean research tax}} + z_{dit})] dt$$

$$s.t. \quad y_{it} = Y_t \left(\frac{p_{it}}{P_t}\right)^{-\sigma}, \qquad y_{it} = \left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\dot{A}_{cit} = \eta_c z_{cit} A_{cit}^{\alpha} A_{dit}^{\gamma}, \quad \dot{A}_{dit} = \eta_d z_{dit} A_{dit}^{\alpha} A_{cit}^{\gamma}$$

Government:

Net taxes T_t collected from the firm are distributed as lump-sum transfers to consumers.

Definition. *A decentralized equilibrium is given by sequences of taxes and subsidies* ($\tau_{cost,t}$, $\tau_{br,t}$, $\tau_{rev,t}$, $\tau_{cl,t}$), *wages* (w_t), *prices for varieties* (p_{it}), *interest rates* (r_t), *demands for machines* (x_{cit}, x_{dit}), *demands for varieties* (y_{it}) *equal to the consumption of varieties* (c_{it}) *and use of machines, demands for researchers* (z_{cit}, z_{dit}), *innovation* (A_{cit}, A_{dit}), *and carbon concentration* (S_t) *such that, in each period* t*:*

- The set of (p_{it}, $x_{cit}, x_{dit}, z_{cit}, z_{dit}$) maximizes profits by the producer of variety i;

- representative agent is maximising utility taking r_t, w_t, p_{it} as given;
- the wage w_t and the prices p_{it} clear the labor and varieties markets respectively;
- the interest rates r_t clears the asset market;
- the evolution of A_{cit}, A_{dit}, S_t is given by:
  - $\dot{A}_{cit} = \eta_c z_{cit} A_{cit}^{\alpha} A_{dit}^{\gamma}$,
  - $\dot{A}_{dit} = \eta_d z_{dit} A_{dit}^{\alpha} A_{cit}^{\gamma}$,
  - $\dot{S}_t = \int_0^1 (1-\delta) A_{dit}^{(1-\beta)} x_{dit}^{\beta} di$

3.1 Results for decentralized equilibrium

First, in the decentralized equilibrium without taxes, compared with the social planner's allocation, there are fewer machines and thus lower output, a lower stock of clean innovation, and a higher stock of dirty innovation. This section summarizes the main results for government policies in the model's decentralized equilibrium. Below, I show the graphs for tax policies in the stable equilibrium, while the graphs for the unstable equilibrium with the same policy implications appear in Appendix A.5. In all figures, S/Y is the balanced-growth-path stock of emissions over output.

1. *A rise in revenue subsidy ($\tau_{rev,t} > 1$) increases the number of machines used, output, and consumption, while also increasing the level of emissions; emissions per unit of output do not change, because the subsidy changes neither A_{cit}/A_{dit} nor x_{cit}/x_{dit}.*

2. *Increase in dirty input tax ($\tau_{br,t} > 1$) reduces emissions.* Raising $\tau_{br,t}$ from 0.95 to 1.05 lowers emissions per unit of output by 5.9% when $\eta_c > \eta_d$ and by 4.9% when $\eta_c < \eta_d$.

3. *Increase in clean research tax ($\tau_{cl,t} > 1$) increases emissions.* Raising $\tau_{cl,t}$ from 0.95 to 1.05 raises emissions per unit of output by 15.6% when $\eta_c > \eta_d$ and by 12.8% when $\eta_c < \eta_d$.

4. *Increase in the cost of capital $\tau_{cost,t} > 1$ amplifies the existing asymmetry in clean versus dirty research productivity.*

   In contrast to the dirty input tax and a clean research subsidy, the direction of change from the policy depends on the relationship between the innovation productivities η_c and η_d. If there is more advancement in the clean innovation ($\eta_c > \eta_d$), an increase

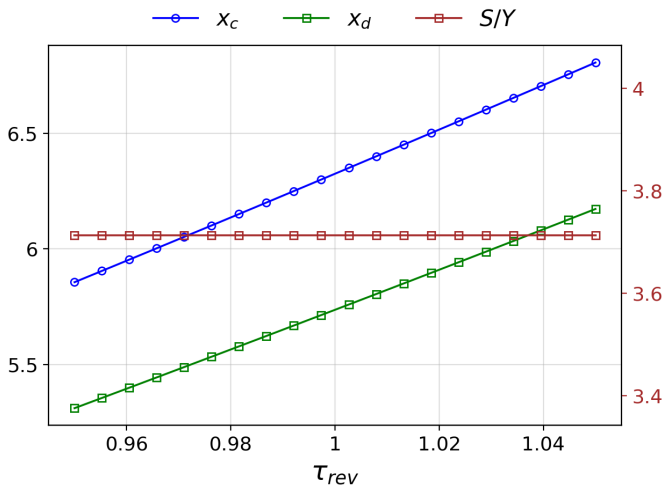
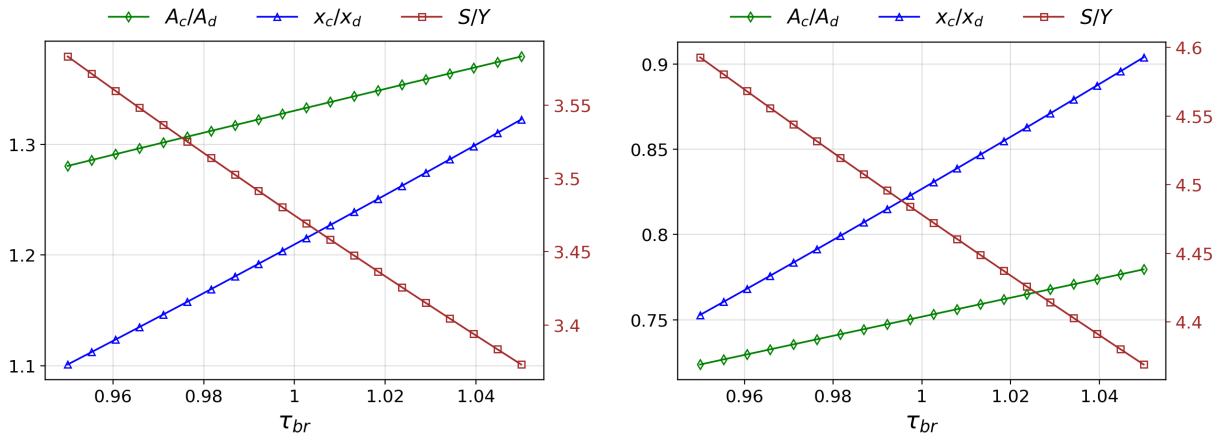


Figure 4. Change in revenue subsidy
The figure represents comparative statics for the change of $\tau_{rev,t}$ for the stable equilibrium, $\eta_c = 0.105$, $\eta_d = 0.1$. x_c and x_d here are detrended demands for machines (x_{cit} and x_{dit} are divided by $L_t^{\frac{1}{1-\alpha-\gamma}}$).



$\eta_c > \eta_d$ $\eta_c < \eta_d$

Figure 5. Change in tax on use of dirty machines
The figure represents comparative statics for the change of $\tau_{br,t}$ for the stable equilibrium; $\eta_c/\eta_d = 1.1$ (left) and $1/1.1$ (right).

in the cost of capital makes the ratio of productivities $\left(\frac{A_{cit}}{A_{dit}}\right)$ higher; thus, firms focus more on the clean innovation when the cost of capital becomes higher. Therefore, an ESG investing strategy that alters the cost of capital of the firm should be based not just on the ESG score or emissions intensity, but also on the degree of substitution of clean and dirty production of the firm, on innovation advancement of these two sectors, and the remaining structure of the innovation process.

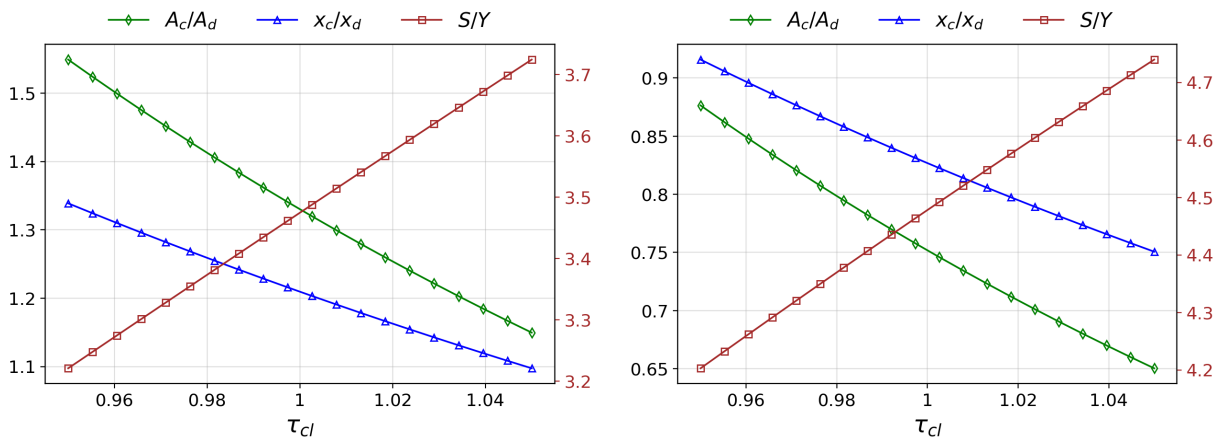


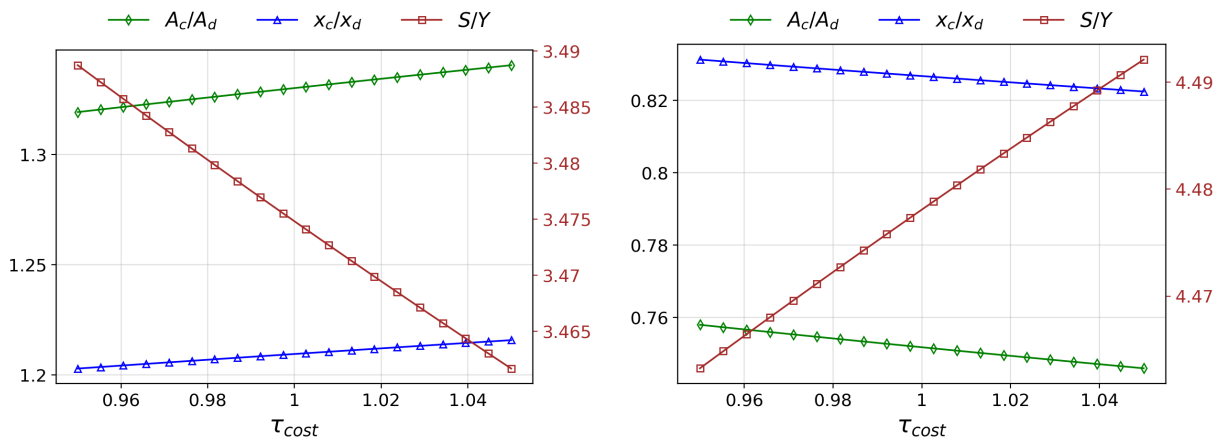
Figure 6. Change in clean research tax
The figure represents comparative statics for the change of $\tau_{cl,t}$ for the stable equilibrium; $\eta_c/\eta_d = 1.1$ (left) and $1/1.1$ (right).

The cost of capital affects the balanced growth path only through the cross-sector spillover: with $\gamma = 0$, $\tau_{cost,t}$ drops out of the equation that pins down A_{cit}/A_{dit} (Appendix A.4). With $\gamma > 0$, part of the value of each technology is the help it gives to future research in the other sector. A higher discount rate cuts this future benefit, which matters most for the lagging technology, so the economy moves further toward the sector with more productive research. The effect is quantitatively small: with $\eta_c/\eta_d = 1.1$, raising $\tau_{cost,t}$ from 0.95 to 1.05 raises A_{cit}/A_{dit} from 1.32 to 1.34 and lowers emissions per unit of output by 0.8%, while the same change in $\tau_{br,t}$ lowers emissions per unit of output by 5.9%. The cost of capital also affects whether the mixed BGP is stable: it raises the effective discount rate in the equilibrium conditions, narrowing the range of ε for which the symmetric BGP is stable when $\gamma > 0$. With $\gamma = 1/12$, the value of ε above which specialized BGPs appear falls from 10 to 9.5 when $\tau_{cost,t}$ rises to 1.05, and a global solution confirms that at $\tau_{cost,t} = 1.5$ an economy with $\varepsilon = 8.5$ that would otherwise stay mixed specializes in one technology (Appendix A.6.7).

A summary of the results is presented in the following table:

Set of taxes matching the social planner's solution ($\theta = 0.03$, $\eta_c = \eta_d = 0.1$) on the balanced growth path is the following:

We see that the revenue subsidy compensates for the monopoly distortion and equals the markup over marginal cost. The dirty input tax corrects the static use of dirty machines. A clean research subsidy of about 2% is also needed because firms value dirty knowledge at



$\eta_c > \eta_d$ $\eta_c < \eta_d$

Figure 7. Increase in the cost of capital
The figure represents comparative statics for the change of $\tau_{cost,t}$ for the stable equilibrium; $\eta_c/\eta_d = 1.1$ (left) and $1/1.1$ (right).

Taxes\Variable	$\left(\frac{A_{cit}}{A_{dit}}\right)^*$	$\left(\frac{x_{cit}}{x_{dit}}\right)^*$	S/Y
$\uparrow$ in revenue subsidy $\tau_{rev,t}$	–	–	–
$\uparrow$ in cost of capital $\tau_{cost,t}$	$\uparrow$ $(\eta_c > \eta_d)$	$\uparrow$ $(\eta_c > \eta_d)$	$\downarrow$ $(\eta_c > \eta_d)$
$\uparrow$ in dirty input tax $\tau_{br,t}$	$\uparrow$	$\uparrow$	$\downarrow$
$\uparrow$ in clean research tax $\tau_{cl,t}$	$\downarrow$	$\downarrow$	$\uparrow$

Table 2. Summary of the results of the decentralized equilibrium. S/Y is emissions per unit of output; the revenue subsidy raises the levels of output and emissions but leaves S/Y unchanged.

$$\tau_{rev,t} = 4/3 = \tfrac{\sigma}{\sigma-1}, \qquad \tau_{br,t} = 1.03$$
$$\tau_{cost,t} = 1, \qquad \tau_{cl,t} = 0.98$$

its private value, ignoring the emissions this knowledge generates. On the balanced growth path, $\tau_{cost,t}$ and $\tau_{cl,t}$ enter only one equation, so they cannot be set separately: any $\tau_{cost,t}$ between 0.9 and 1.5 replicates the planner's allocation together with a matching $\tau_{cl,t}$ between 0.977 and 0.985. A cost-of-capital wedge is therefore not needed; I set $\tau_{cost,t} = 1$, at which point the firm discounts at the planner's rate $\rho - n + g_A$.

4 Next steps

First, as the next step, I am planning to make a richer model with explicitly modeled capital instead of machines, labor employed not just in research, but also in the production sector, non-symmetric firms, and the ESG preferences of the investors. It will allow me to

structurally estimate the interplay between government regulation and ESG investing and test whether a change in the stringency of regulation will make ESG funds reward brown firms more for green patenting. Second, I want to structurally estimate the parameters of the innovation process to provide data-driven policy recommendations, as the stability of equilibria and policy implications depend on the innovation process and cross-sector spillovers.

Estimation of the innovation process will look as follows. The dynamic innovation possibilities frontier of the current model is:

$$\frac{\dot{A}_{jit}}{A_{jit}} = \eta_{jit} z_{jit} A_{jit}^{\alpha-1} A_{\{-j\}it}^{\gamma}$$

where A_{jit} is the stock of patents, i is the company's identifier, $j = \{c, d\}$, z_{jit} is a research effort, η_{jit} is an innovation productivity (spillovers).

I will adjust the process in line with Acemoglu (2023):

$$\frac{\dot{A}_{jit}}{A_{jit}} = \xi_{jit} \Gamma_{jt} \eta_{jit} z_{jit} A_{jit}^{\alpha-1} A_{\{-j\}it}^{\gamma}$$

where Γ_{jt} is a random term, and ξ_{jit} is a term orthogonal to everything else.

I am planning to run the following regression in line with Acemoglu (2023): to estimate own- and cross-spillovers of clean and dirty innovation:

$$ln\left(\frac{\Delta A_{jit}}{A_{jit}}\right) = \Gamma_{jt} + \eta_{jit} + ln(z_{jit}) + (\alpha - 1) ln(A_{jit}) + \gamma ln(A_{\{-j\}it} + \xi_{jit}$$

I will instrument labor as $ln(z_{jit}) = \kappa ln(\tau_{jit}) + \lambda \Delta ln(\tau_{jit}) + \nu_t$, where τ_{jit} are fuel prices/ market sizes / policies.

The **data** I am planning to use is the following:

- *patents*: PATSTAT, Orbis Intellectual Property, Google Patents; classification: IPC, CPC, OECD, Aghion et al. (2016), USPTO.
- *energy use and prices*: US energy information administration, Our World in Data.
- *firms characteristics and R&D expenses*: Compustat, Orbis, CRSP.

To evaluate whether a change in the stringency of regulation will make ESG funds reward brown firms more for green patenting I will use an additional data:

- *emissions*: TruCost S&P, Environmental Protection Agency (EPA), TRI (Toxic Release Inventory).

- *ESG rankings*: S&P Trucost, Sustainalytics, MSCI, ISS, Thomson Reuters Refinitiv ASSET4, FTSE Russell.
- *firms characteristics and R&D expenses*: Compustat, Orbis, CRSP.
- fraction of *impact investors*: SIF report on US Sustainable and Impact Investing Trends to compute the union of all stocks held by ESG funds, SEC Form N-PORT, list of "Sustainable Investment Mutual Funds and ETFs" offered by institutional member firms of "The Forum for Sustainable and Responsible Investing".
- *relative sectoral demand at the household level*: U.S. Consumer Expenditure Survey (CEX) and MRI's Survey of the American Consumer (MRI), which contains the composition of household consumption expenditures on different final goods.

5 Conclusion

Brown companies can and do produce green innovations; thus, it is important to carefully evaluate the impact of sustainable investing by individual investors and central banks on the production and innovation of brown and mixed companies in a general equilibrium model. A multiproduct firm setting with directed technical change is crucial for evaluating the effect of ESG investing, and the parsimonious model developed in this paper can become a foundation for future research on this topic. I show that an ESG investing strategy should be based not only on the ESG score or emissions intensity, but also on the degree of substitution between clean and dirty production, the innovation advancement of these two sectors, and the remaining structure of the innovation process. I find that, in line with the literature, a rise in dirty input tax and a rise in a clean research subsidy reduce emissions, and, importantly, an increase in the cost of capital amplifies the existing asymmetry in clean versus dirty research productivity. In contrast to the dirty input tax and clean research subsidy, the effect of the sustainable investing strategy depends on the relationship between innovation productivities in the clean and dirty innovation processes. If dirty innovation advances more, an increase in the cost of capital lowers the clean-to-dirty productivity ratio, and firms focus more on dirty innovation. Thus, increasing the cost of capital for companies that initially advance dirty technology can lead to a "Green Paradox," where higher costs make emissions grow and disrupt the green transition. In the current calibration, this effect is several times smaller than the effect of a dirty input tax of the same size: a 10% change in the cost of capital moves emissions per unit of output by 0.8%, against 5.9% for the dirty input tax. A higher cost of capital can also tip a mixed economy into specialization when spillovers are positive,

and the two outputs are highly substitutable. In future steps, I will structurally estimate the innovation process and enrich the model with more realistic investor ESG preferences to match the observed effect on the cost of capital in the data.

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A Appendix

A.1 Carbon cycle and damage function

Carbon concentration: from Nordhaus (1994), Golosov et al. (2014), Acemoglu et al. (2016):

$$S_t = \int_0^{t-T} (1-\delta_l) E_{t-l} dl$$

- $t = T$ – the first date when emission E_{t-l} started
- $\delta_l = (1-\phi_p)(1-\phi_0 e^{-\phi l})$ – share of carbon emitted l years ago that has left the atmosphere, so $1-\delta_l$ is the share still in it;
- $\phi_p \in (0,1)$ – the share of emissions that remains permanently in the atmosphere;
- $(1-\phi_p)\phi_0$ – the fraction of the transitory component that remains in the first period;
- $\phi \in (0,1)$ – the rate of decay of carbon concentration over time.

Damage function $D(S_t)$ (alternative - disutility of emissions):
Affected output: $(1-D(S_t))Y_t = e^{-\zeta(S_t-\bar{S})}Y_t$,
where $\bar{S}$ is the pre-industrial atmospheric CO_2 concentration.

A.2 Classification of patents

The table below reports the Cooperative Patent Classification (CPC) and International Patent Classification (IPC) used to determine clean and dirty types of innovation in the car industry, similar to Aghion et al. (2016). The Orbis data used in Figure 1 report only the main IPC code of each patent publication, so Figure 1 uses the IPC codes; the grey codes are counted as dirty there.

Table 3. Patent classification codes for car industry

Clean	
Y02T10/60 (CPC)	Other road transportation technologies with climate change mitigation effect
Y02T10/70 (CPC)	Energy storage for electromobility
Y02T90/10 (CPC)	Technologies related to electric vehicle charging
Y02T90/34 (CPC)	Fuel cell powered electric vehicles
Y02T90/42 (CPC)	Hydrogen as fuel for road transportation
Electric vehicles	
B60L 11 (IPC)	Electric propulsion with power supplied within the vehicle
B60L 50, 53, 55, 58 (IPC)	Codes that replaced B60L 11 in IPC 2019: electric propulsion with power supplied within the vehicle, charging, vehicle-to-grid, monitoring of batteries
B60L 3 (IPC)	Electric devices on electrically-propelled vehicles for safety purposes; Monitoring operating variables, e.g. speed, deceleration, power consumption
B60L 15 (IPC)	Methods, circuits, or devices for controlling the traction- motor speed of electrically-propelled vehicles
B60K 1 (IPC)	Arrangement or mounting of electrical propulsion units
B60W 10/08, 24, 26 (IPC)	Conjoint control of vehicle sub-units of different type or different function / including control of electric propulsion units, e.g. motors or generators / including control of energy storage means / for electrical energy, e.g. batteries or capacitors
Hybrid vehicles	
B60K 6 (IPC)	Arrangement or mounting of plural diverse prime-movers for mutual or common propulsion, e.g. hybrid propulsion systems comprising electric motors and internal combustion engines
B60W 20 (IPC)	Control systems specially adapted for hybrid vehicles, i.e. vehicles having two or more prime movers of more than one type, e.g. electrical and internal combustion motors, all used for propulsion of the vehicle
B60L 7/10–7/18 (IPC)	Dynamic electric regenerative braking
B60L 7/20 (IPC)	Braking by supplying regenerated power to the prime mover of vehicles comprising engine-driven generators
Hydrogen vehicles / fuel cells	
B60W 10/28 (IPC)	Conjoint control of vehicle sub-units of different type or different function; including control of fuel cells
B60L 11/18 (IPC)	Electric propulsion with power supplied within the vehicle - using power supplied from primary cells, secondary cells, or fuel cells
H01M 8 (IPC)	Fuel cells; Manufacture thereof

Grey	
Y02T10/10 (CPC)	Climate change mitigation technologies related to fuel injection
Y02T10/20 (CPC)	Climate change mitigation technologies related to exhaust after treatment
Y02T10/40 (CPC)	Climate change mitigation technologies related to engine Management Systems
Y02T10/50 (CPC)	Climate change mitigation technologies related to Intelligent Control Systems
F02M39-71 (IPC)	Fuel injection apparatus
F02M3/02-05 (IPC)	Idling devices for carburettors preventing flow of idling fuel
F02M23 (IPC)	Apparatus for adding secondary air to fuel-air mixture
F02M25 (IPC)	Engine-pertinent apparatus for adding non-fuel substances or small quantities of secondary fuel to combustion-air, main fuel, or fuel-air mixture
F02D41 (IPC)	Electrical control of supply of combustible mixture or its constituents
F02B47/06 (IPC)	Methods of operating engines involving adding non-fuel substances or anti-knock agents to combustion air, fuel, or fuel-air mixtures of engines, the substances including non-airborne oxygen
Dirty	
F02 (CPC)	Combustion Engines
F02B (IPC)	Internal-combustion piston engines; combustion engines in general
F02D (IPC)	Controlling combustion engines
F02F (IPC)	Cylinders, pistons, or casings for combustion engines; arrangement of sealings in combustion engines
F02M (IPC)	Supplying combustion engines with combustible mixtures or constituents thereof
F02N (IPC)	Starting of combustion engines
F02P (IPC)	Ignition (other than compression ignition) for internal-combustion engines
Other	
B60 (CPC)	Vehicles in General

A.3 Detailed solution of the model: social planner's problem

A.3.1 Assuming no internalization of emissions by social planner:

This section is used for more tractability and ability to compare the social planner's allocation when there is no disutility of emissions $\theta = 0$ to the decentralized equilibrium with no taxes, where the only difference is the monopolistic competition of the firm and a resulting markup over the marginal cost.

Hamiltonian:

$$\begin{aligned}
H =& \frac{\sigma}{\sigma-1} log\left(\int_0^1 (C_{it})^{\frac{\sigma-1}{\sigma}}\, di\right) - \log(L_t) - \theta log(S_t) + \\
&+\lambda_t \left[\int_0^1 \left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} di - \right. \\
&\left. -\psi \int_0^1 x_{cit} di - \psi \int_0^1 x_{dit} di - \int_0^1 C_{it} di\right] \\
&+\mu_{1t}[\eta_c z_{cit} A_{cit}^{\alpha} A_{dit}^{\gamma}] + \mu_{2t}[\eta_d z_{dit} A_{dit}^{\alpha} A_{cit}^{\gamma}] + \kappa_t \left[L_t - \int_0^1 z_{cit} di - \int_0^1 z_{dit} di\right]
\end{aligned}$$

FOCs:

$$\begin{aligned}
&H_{C_{it}} = 0, \\
&H_{x_{cit}} = 0, \\
&H_{x_{dit}} = 0, \\
&H_{z_{cit}} = 0, \\
&H_{z_{dit}} = 0 \\
&H_{A_{cit}} = (\rho - n)\mu_{1t} - \dot{\mu_{1t}} \\
&H_{A_{dit}} = (\rho - n)\mu_{2t} - \dot{\mu_{2t}}
\end{aligned}$$

Derivation of the inner solution of the Balanced growth path

$$\begin{aligned}
H_{C_{it}} = 0: &\quad \frac{C_{it}^{-1/\sigma}}{\int_0^1 C_{it}^{\frac{\sigma-1}{\sigma}} di} \equiv \frac{C_{it}^{-1/\sigma}}{C_t^{\frac{\sigma-1}{\sigma}}} = \lambda_t \\
H_{x_{cit}} = 0: &\quad \lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi\right] = 0 \\
H_{x_{dit}} = 0: &\quad \lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi\right] = 0
\end{aligned}$$

$$H_{z_{cit}} = 0: \quad \mu_{1t}\eta_c A_{cit}^{\alpha} A_{dit}^{\gamma} = \kappa_t$$

$$H_{z_{dit}} = 0: \quad \mu_{2t}\eta_d A_{cit}^{\gamma} A_{dit}^{\alpha} = \kappa_t \;\rightarrow\; \mu_{1t} = \mu_{2t}\frac{\eta_d}{\eta_c} A_{cit}^{\gamma-\alpha} A_{dit}^{\alpha-\gamma}, \qquad \mu_{2t} = \mu_{1t}\frac{\eta_c}{\eta_d} A_{dit}^{\gamma-\alpha} A_{cit}^{\alpha-\gamma}$$

$$H_{A_{cit}} = (\rho - n)\mu_{1t} - \dot{\mu_{1t}}:$$

$$\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}} \right] +$$

$$+ \mu_{1t}\alpha\eta_c z_{cit} A_{cit}^{\alpha-1} A_{dit}^{\gamma} + \mu_{2t}\gamma\eta_d z_{dit} A_{cit}^{\gamma-1} A_{dit}^{\alpha} = (\rho - n)\mu_{1t} - \dot{\mu}_{1t}$$

$$H_{A_{dit}} = (\rho - n)\mu_{2t} - \dot{\mu_{2t}}:$$

$$\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}} \right] +$$

$$+ \mu_{1t}\gamma\eta_c z_{cit} A_{cit}^{\alpha} A_{dit}^{\gamma-1} + \mu_{2t}\alpha\eta_d z_{dit} A_{cit}^{\gamma} A_{dit}^{\alpha-1} = (\rho - n)\mu_{2t} - \dot{\mu}_{2t}$$

We see that $A_{cit} = A_{ct}, x_{cit} = x_{ct}$ etc.

Recalling an innovation process:

$$\frac{\dot{A}_{cit}}{A_{cit}} = \eta_c z_{cit} A_{cit}^{\alpha-1} A_{dit}^{\gamma},$$

$$\frac{\dot{A}_{dit}}{A_{dit}} = \eta_d z_{dit} A_{dit}^{\alpha-1} A_{cit}^{\gamma}$$

For the existence of the BGP (constant growth rates of all variables) we need:

$$\frac{\dot{g}_{A_c}}{g_{A_c}} = g_{z_c} + (\alpha - 1)g_{A_c} + \gamma g_{A_d} = 0$$

$$\frac{\dot{g}_{A_d}}{g_{A_d}} = g_{z_d} + (\alpha - 1)g_{A_d} + \gamma g_{A_c} = 0$$

$BGP:$

$$g_{A_d} = g_{A_c} = n\frac{1}{1-\alpha-\gamma} = g_y = g_x = g_A$$

$$z_{cit} = g_{A_c}\frac{1}{\eta_c A_{cit}^{\alpha-1} A_{dit}^{\gamma}}$$

$$z_{dit} = g_{A_d}\frac{1}{\eta_d A_{cit}^{\gamma} A_{dit}^{\alpha-1}}$$

Thus, at the balanced growth path growth rates A_{cit} and A_{dit} are the same, and other growth rates are $g_c = g_x = g_A$.

Simplifying, we get:

- $\dfrac{C_{it}^{-1/\sigma}}{C_t^{\frac{\sigma-1}{\sigma}}} = \lambda_t$

- $\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}+(A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}\beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1}-\psi=0$
- $\left(\frac{x_{cit}}{x_{dit}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}-1}\left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}=1$
- $\mu_{1t}\frac{\eta_c}{\eta_d}\left(\frac{A_{cit}}{A_{dit}}\right)^{\alpha-\gamma}=\mu_{2t}$
- $\mu_{1t}=\dfrac{\lambda_t\left[\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}+(A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}(1-\beta)A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1}x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}}\right]}{(\rho-n)-\frac{\dot{\mu}_{1t}}{\mu_{1t}}-\eta_c A_{cit}^{\alpha-1}A_{dit}^{\gamma}[\alpha z_{cit}+\gamma z_{dit}]}$
- $\mu_{2t}=\dfrac{\lambda_t\left[\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}+(A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}(1-\beta)A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1}x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}}\right]}{(\rho-n)-\frac{\dot{\mu}_{2t}}{\mu_{2t}}-\eta_d A_{cit}^{\gamma}A_{dit}^{\alpha-1}[\gamma z_{cit}+\alpha z_{dit}]}$
- $z_{cit}=g_{A_c}\dfrac{1}{\eta_c A_{cit}^{\alpha-1}A_{dit}^{\gamma}}$
- $z_{dit}=g_{A_d}\dfrac{1}{\eta_d A_{cit}^{\gamma}A_{dit}^{\alpha-1}}$
- $z_{cit}+z_{dit}=L_t$

$\frac{\dot{\mu}_{1t}}{\mu_{1t}}=\left(-1+1-\frac{\varepsilon-1}{\varepsilon}+(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1+\beta\frac{\varepsilon-1}{\varepsilon}\right)g_A=-g_A,$
$\frac{\dot{\mu}_{2t}}{\mu_{2t}}=-g_A.$
Substituting μ_{1t},μ_{2t} and using the expressions for the growth rates, we get:

- $\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}+(A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}\beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1}-\psi=0$
- $\left(\frac{x_{cit}}{x_{dit}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}-1}\left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}=1$
- $\dfrac{\left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1}\left(\frac{x_{cit}}{x_{dit}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}}}{\rho-n+g_A(1-\alpha)-g_A\frac{\eta_c}{\eta_d}\left(\frac{A_{cit}}{A_{dit}}\right)^{\alpha-1-\gamma}\gamma}\frac{\eta_c}{\eta_d}\left(\frac{A_{cit}}{A_{dit}}\right)^{\alpha-\gamma}=$
$=\dfrac{1}{\rho-n+g_A(1-\alpha)-g_A\frac{\eta_d}{\eta_c}\left(\frac{A_{cit}}{A_{dit}}\right)^{\gamma-\alpha+1}\gamma}$
- $g_{A_c}\dfrac{1}{\eta_c A_{cit}^{\alpha-1}A_{dit}^{\gamma}}+g_{A_d}\dfrac{1}{\eta_d A_{cit}^{\gamma}A_{dit}^{\alpha-1}}=L_t$ (market clearing condition for researchers).

From here I numerically solve for detrended (divided by $L_t^{\frac{1}{1-\alpha-\gamma}}$) $x_{cit},x_{dit},A_{cit},A_{dit}$.

Derivation of the corner solution of the Balanced growth path Let's consider the clean corner, where $z_{cit} = z_{ct} = L_t,\ z_{dit} = z_{dt} = 0$ (symmetrically for the dirty corner).

$$H_{C_{it}} = 0: \quad \frac{C_{it}^{-1/\sigma}}{\int_0^1 C_{it}^{\frac{\sigma-1}{\sigma}} di} \equiv \frac{C_{it}^{-1/\sigma}}{C_t^{\frac{\sigma-1}{\sigma}}} = \lambda_t$$

$$H_{x_{cit}} = 0: \quad \lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi \right] = 0$$

$$H_{x_{dit}} = 0: \quad \lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi \right] = 0$$

$$H_{z_{cit}} = 0: \quad \mu_{1t}\eta_c A_{cit}^{\alpha} A_{dit}^{\gamma} > \mu_{2t}\eta_d A_{cit}^{\gamma} A_{dit}^{\alpha}$$

$$H_{A_{cit}} = (\rho - n)\mu_{1t} - \dot{\mu_{1t}}:$$

$$\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}} \right] +$$

$$+ \mu_{1t}\alpha\eta_c z_{cit} A_{cit}^{\alpha-1} A_{dit}^{\gamma} + \mu_{2t}\gamma\eta_d z_{dit} A_{cit}^{\gamma-1} A_{dit}^{\alpha} = (\rho - n)\mu_{1t} - \dot{\mu}_{1t}$$

$$H_{A_{dit}} = (\rho - n)\mu_{2t} - \dot{\mu_{2t}}:$$

$$\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}} \right] +$$

$$+ \mu_{1t}\gamma\eta_c z_{cit} A_{cit}^{\alpha} A_{dit}^{\gamma-1} + \mu_{2t}\alpha\eta_d z_{dit} A_{cit}^{\gamma} A_{dit}^{\alpha-1} = (\rho - n)\mu_{2t} - \dot{\mu}_{2t}$$

Using that $z_{cit} = z_{ct} = L_t,\ z_{dit} = z_{dt} = 0$:

- $\dfrac{C_{it}^{-1/\sigma}}{C_t^{\frac{\sigma-1}{\sigma}}} = \lambda_t$
- $\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi \right] = 0$
- $\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi \right] = 0$
- $\mu_{1t} \dfrac{\eta_c}{\eta_d} \left(\dfrac{A_{cit}}{A_{dit}} \right)^{\alpha-\gamma} > \mu_{2t}$
- $\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}} \right] +$
  $+ \mu_{1t}\alpha\eta_c z_{cit} A_{cit}^{\alpha-1} A_{dit}^{\gamma} = (\rho - n)\mu_{1t} - \dot{\mu}_{1t}$
- $\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}} \right] +$
  $+ \mu_{1t}\gamma\eta_c z_{cit} A_{cit}^{\alpha} A_{dit}^{\gamma-1} = (\rho - n)\mu_{2t} - \dot{\mu}_{2t}$

Simplifying further:

- $\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi \right] = 0$
- $\left(\frac{x_{cit}}{x_{dit}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}-1} \left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} = 1$
- $\mu_{1t} \frac{\eta_c}{\eta_d} \left(\frac{A_{cit}}{A_{dit}}\right)^{\alpha-\gamma} > \mu_{2t}$
- $\mu_{1t} = \dfrac{\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}} \right]}{\rho - n - \frac{\dot{\mu}_{1t}}{\mu_{1t}} - \alpha g_{A_c}}$
- $\mu_{2t} = \dfrac{\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) \right] \left(A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}} + \frac{A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}}}{\rho-n-\frac{\dot{\mu}_{1t}}{\mu_{1t}}-\alpha g_{A_c}} \gamma g_{A_c} \frac{A_{cit}}{A_{dit}} \right)}{\rho - n - \frac{\dot{\mu}_{2t}}{\mu_{2t}}}$

The resulting system is:

- $\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi \right] = 0$
- $\left(\frac{x_{cit}}{x_{dit}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}-1} \left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} = 1$
- $\dfrac{\left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} \left(\frac{x_{cit}}{x_{dit}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}}}{\rho - n - \frac{\dot{\mu}_{1t}}{\mu_{1t}} - \alpha g_{A_c}} \frac{\eta_c}{\eta_d} \left(\frac{A_{cit}}{A_{dit}}\right)^{\alpha-\gamma} > \dfrac{1 + \frac{\left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} \left(\frac{x_{cit}}{x_{dit}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}}}{\rho-n-\frac{\dot{\mu}_{1t}}{\mu_{1t}}-\alpha g_{A_c}} \gamma g_{A_c} \frac{A_{cit}}{A_{dit}}}{\rho - n - \frac{\dot{\mu}_{2t}}{\mu_{2t}}}$

Substituting $\left(\frac{x_{cit}}{x_{dit}}\right)$ to the inequality:

- $\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi \right] = 0$
- $\left(\frac{x_{cit}}{x_{dit}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}-1} \left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} = 1$
- $\dfrac{\left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1-\frac{\beta\frac{\varepsilon-1}{\varepsilon}(1-\beta)\frac{\varepsilon-1}{\varepsilon}}{\beta\frac{\varepsilon-1}{\varepsilon}-1}+\alpha-\gamma}}{\rho - n - \frac{\dot{\mu}_{1t}}{\mu_{1t}} - \alpha g_{A_c}} \frac{\eta_c}{\eta_d} > \dfrac{1 + \frac{\left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1-\frac{\beta\frac{\varepsilon-1}{\varepsilon}(1-\beta)\frac{\varepsilon-1}{\varepsilon}}{\beta\frac{\varepsilon-1}{\varepsilon}-1}+1}}{\rho-n-\frac{\dot{\mu}_{1t}}{\mu_{1t}}-\alpha g_{A_c}} \gamma g_{A_c}}{\rho - n - \frac{\dot{\mu}_{2t}}{\mu_{2t}}}$

when $z_{cit} = L_t, z_{dit} = 0, \frac{A_{cit}}{A_{dit}} \to \infty$
For the inequality to hold:

$$\begin{bmatrix} (1-\beta)^{\frac{\varepsilon-1}{\varepsilon}} - 1 - \frac{\beta^{\frac{\varepsilon-1}{\varepsilon}}(1-\beta)^{\frac{\varepsilon-1}{\varepsilon}}}{\beta^{\frac{\varepsilon-1}{\varepsilon}}-1} + \alpha - \gamma > 0, & \gamma \leq 0 \\ or & \\ (1-\beta)^{\frac{\varepsilon-1}{\varepsilon}} - 1 - \frac{\beta^{\frac{\varepsilon-1}{\varepsilon}}(1-\beta)^{\frac{\varepsilon-1}{\varepsilon}}}{\beta^{\frac{\varepsilon-1}{\varepsilon}}-1} + \alpha - \gamma \leq 0, & \gamma < 0, \\ (1-\beta)^{\frac{\varepsilon-1}{\varepsilon}} - 1 - \frac{\beta^{\frac{\varepsilon-1}{\varepsilon}}(1-\beta)^{\frac{\varepsilon-1}{\varepsilon}}}{\beta^{\frac{\varepsilon-1}{\varepsilon}}-1} + 1 > 0 & \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon > \frac{1-\beta(\alpha-\gamma)}{(1-\beta)(\alpha-\gamma)} = 1 + \frac{1-(\alpha-\gamma)}{(1-\beta)(\alpha-\gamma)}, & \gamma \leq 0 \\ or & \\ \varepsilon \leq \frac{1-\beta(\alpha-\gamma)}{(1-\beta)(\alpha-\gamma)} = 1 + \frac{1-(\alpha-\gamma)}{(1-\beta)(\alpha-\gamma)}, & \gamma < 0, \quad \varepsilon > 1 \end{bmatrix}$$

This is a condition shown in the main text of the paper. The second case requires $\gamma < 0$ strictly: with $\gamma = 0$ the term multiplied by γ on the right side vanishes, and the inequality holds only if the exponent on the left side is positive. With $\gamma > 0$ the right side grows faster than the left side, because the difference of the two exponents is $\alpha - \gamma - 1 < 0$, so a corner path cannot satisfy the inequality.

A.3.2 Assuming full internalization of emissions by social planner:

Hamiltonian:

$$\begin{aligned} H =& \frac{\sigma}{\sigma-1} log\left(\int_0^1 (C_{it})^{\frac{\sigma-1}{\sigma}} di\right) - \log(L_t) - \theta log(S_t) + \\ &+ \lambda_t \left[\int_0^1 \left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} di - \right. \\ &\left. - \psi \int_0^1 x_{cit} di - \psi \int_0^1 x_{dit} di - \int_0^1 C_{it} di\right] \\ &+ \mu_{1t}[\eta_c z_{cit} A_{cit}^{\alpha} A_{dit}^{\gamma}] + \mu_{2t}[\eta_d z_{dit} A_{dit}^{\alpha} A_{cit}^{\gamma}] + \kappa_t \left[L_t - \int_0^1 z_{cit} di - \int_0^1 z_{dit} di\right] \\ &+ \xi_t \left[\int_0^1 (1-\delta) A_{dit}^{(1-\beta)} x_{dit}^{\beta} di\right] \end{aligned}$$

FOCs:

$$\begin{aligned} H_{C_{it}} &= 0, \\ H_{x_{cit}} &= 0, \\ H_{x_{dit}} &= 0, \\ H_{z_{cit}} &= 0, \end{aligned}$$

$$H_{z_{dit}} = 0$$
$$H_{A_{cit}} = (\rho - n)\mu_{1t} - \dot{\mu_{1t}}$$
$$H_{A_{dit}} = (\rho - n)\mu_{2t} - \dot{\mu_{2t}}$$
$$H_{S_t} = (\rho - n)\xi_t - \dot{\xi}_t$$

Derivation of the inner solution of the Balanced growth path

$$H_{C_{it}} = 0: \quad \frac{C_{it}^{-1/\sigma}}{\int_0^1 C_{it}^{\frac{\sigma-1}{\sigma}} di} \equiv \frac{C_{it}^{-1/\sigma}}{C_t^{\frac{\sigma-1}{\sigma}}} = \lambda_t$$

$$H_{x_{cit}} = 0: \quad \lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi \right] = 0$$

$$H_{x_{dit}} = 0: \quad \lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi \right] +$$
$$+ \xi_t (1-\delta)\beta A_{dit}^{1-\beta} x_{dit}^{\beta-1} = 0$$

$$H_{z_{cit}} = 0: \quad \mu_{1t}\eta_c A_{cit}^{\alpha} A_{dit}^{\gamma} = \kappa_t$$

$$H_{z_{dit}} = 0: \quad \mu_{2t}\eta_d A_{cit}^{\gamma} A_{dit}^{\alpha} = \kappa_t \;\rightarrow\; \mu_{1t} = \mu_{2t}\frac{\eta_d}{\eta_c} A_{cit}^{\gamma-\alpha} A_{dit}^{\alpha-\gamma}, \qquad \mu_{2t} = \mu_{1t}\frac{\eta_c}{\eta_d} A_{dit}^{\gamma-\alpha} A_{cit}^{\alpha-\gamma}$$

$$H_{A_{cit}} = (\rho - n)\mu_{1t} - \dot{\mu_{1t}}:$$
$$\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}} \right] +$$
$$+ \mu_{1t}\alpha\eta_c z_{cit} A_{cit}^{\alpha-1} A_{dit}^{\gamma} + \mu_{2t}\gamma\eta_d z_{dit} A_{cit}^{\gamma-1} A_{dit}^{\alpha} = (\rho - n)\mu_{1t} - \dot{\mu}_{1t}$$

$$H_{A_{dit}} = (\rho - n)\mu_{2t} - \dot{\mu_{2t}}:$$
$$\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}} \right] +$$
$$+ \mu_{1t}\gamma\eta_c z_{cit} A_{cit}^{\alpha} A_{dit}^{\gamma-1} + \mu_{2t}\alpha\eta_d z_{dit} A_{cit}^{\gamma} A_{dit}^{\alpha-1} + \xi_t(1-\delta)(1-\beta)A_{dit}^{-\beta} x_{dit}^{\beta} =$$
$$= (\rho - n)\mu_{2t} - \dot{\mu}_{2t}$$

$$H_{S_t} = (\rho - n)\xi_t - \dot{\xi}_t:$$
$$-\frac{\theta}{S_t} = (\rho - n)\xi_t - \dot{\xi}_t$$
$$\frac{\dot{S}_t}{S_t} = \frac{\int_0^1 (1-\delta) A_{dit}^{(1-\beta)} x_{dit}^{\beta} di}{S_t} = (1-\beta) g_{A_d} + \beta g_{x_d} = g_A$$

Using the expressions for ξ_t, μ_{1t}, μ_{2t}, S_t:

- $\dfrac{C_{it}^{-1/\sigma}}{C_t^{\frac{\sigma-1}{\sigma}}} = \lambda_t$

- $\mu_{1t}\frac{\eta_c}{\eta_d}\left(\frac{A_{cit}}{A_{dit}}\right)^{\alpha-\gamma} = \mu_{2t}$
- $\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi = 0$
- $\lambda_t\left[\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta)A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}}\right] +$
  $+ \mu_{1t}\eta_c A_{cit}^{\alpha-1}A_{dit}^{\gamma}[\alpha z_{cit} + \gamma z_{dit}] = (\rho-n)\mu_{1t} - \dot{\mu}_{1t}$
- $\lambda_t\left[\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta)A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}}\right] +$
  $+ \mu_{2t}\eta_d A_{cit}^{\gamma}A_{dit}^{\alpha-1}[\gamma z_{cit} + \alpha z_{dit}] -$
  $- \frac{\lambda_t(1-\beta)}{A_{dit}x_{dit}^{-1}}\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1} \cdot$
  $\cdot\left(A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1}\right) =$
  $= (\rho-n)\mu_{2t} - \dot{\mu}_{2t}$
- $\frac{\theta g_A}{x_{dit}\lambda_t\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}\left(A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1}\right)} =$
  $= (\rho-n) - \frac{\dot{\xi}_t}{\xi_t}$

  Growth rates:

$$
\begin{aligned}
&g_c = g_x = g_A \\
&g_\xi = -g_c - (1-\beta)g_A - (\beta-1)g_x + \\
&+\left(1-\frac{\varepsilon-1}{\varepsilon}\right)g_A + \left((1-\beta)\frac{\varepsilon-1}{\varepsilon}\right)g_A + \left(\beta\frac{\varepsilon-1}{\varepsilon}-1\right)g_x = \\
&= \left(-1+1-\frac{\varepsilon-1}{\varepsilon}+(1-\beta)\frac{\varepsilon-1}{\varepsilon}+\beta\frac{\varepsilon-1}{\varepsilon}-1\right)g_A = -g_A
\end{aligned}
$$

- $\frac{C_{it}^{-1/\sigma}}{C_t^{\frac{\sigma-1}{\sigma}}} = \lambda_t$
- $\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi = 0$
- $\mu_{1t}\frac{\eta_c}{\eta_d}\left(\frac{A_{cit}}{A_{dit}}\right)^{\alpha-\gamma} = \mu_{2t}$

- $\mu_{1t} = \dfrac{\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}} \right]}{(\rho-n) - \frac{\dot{\mu}_{1t}}{\mu_{1t}} - \eta_c A_{cit}^{\alpha-1} A_{dit}^{\gamma} [\alpha z_{cit} + \gamma z_{dit}]}$
- $\mu_{2t} = \dfrac{\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} A_{dit}^{-1} x_{dit} \right]}{(\rho-n) - \frac{\dot{\mu}_{2t}}{\mu_{2t}} - \eta_d A_{cit}^{\gamma} A_{dit}^{\alpha-1} [\gamma z_{cit} + \alpha z_{dit}]}$
- $\dfrac{\theta g_A \left(\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}} - \psi(x_{cit} + x_{dit}) \right)}{x_{dit} \left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \left(A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} \right)} =$
  $= \rho - n + g_A$

  $\frac{\dot{\mu}_{1t}}{\mu_{1t}} = \left(-1 + 1 - \frac{\varepsilon-1}{\varepsilon} + (1-\beta)\frac{\varepsilon-1}{\varepsilon} - 1 + \beta\frac{\varepsilon-1}{\varepsilon} \right) g_A = -g_A$
  $\frac{\dot{\mu}_{2t}}{\mu_{2t}} = -g_A$

  After further simplification, we get:

- $\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi = 0$
- $\dfrac{A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}}}{\rho - n + g_A(1-\alpha) - g_A \frac{\eta_c}{\eta_d} A_{cit}^{\alpha-1-\gamma} A_{dit}^{\gamma-\alpha+1} \gamma} \dfrac{\eta_c}{\eta_d} \left(\dfrac{A_{cit}}{A_{dit}} \right)^{\alpha-\gamma} =$
  $= \dfrac{A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} A_{dit}^{-1} x_{dit}}{\rho - n + g_A(1-\alpha) - g_A \frac{\eta_d}{\eta_c} A_{cit}^{\gamma-\alpha+1} A_{dit}^{\alpha-1-\gamma} \gamma}$
- $\dfrac{\theta g_A \left(\left(\frac{A_{cit}}{A_{dit}} \right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} \left(\frac{x_{cit}}{x_{dit}} \right)^{\beta\frac{\varepsilon-1}{\varepsilon}} + 1 - \beta \left(\frac{A_{cit}}{A_{dit}} \right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} \left(\frac{x_{cit}}{x_{dit}} \right)^{\beta\frac{\varepsilon-1}{\varepsilon}} \left(1 + \left(\frac{x_{cit}}{x_{dit}} \right)^{-1} \right) \right)}{\left(1 - \left(\frac{A_{cit}}{A_{dit}} \right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} \left(\frac{x_{cit}}{x_{dit}} \right)^{\beta\frac{\varepsilon-1}{\varepsilon}-1} \right)} =$
  $= \rho - n + g_A$
- $g_{A_c} \dfrac{1}{\eta_c A_{cit}^{\alpha-1} A_{dit}^{\gamma}} + g_{A_d} \dfrac{1}{\eta_d A_{cit}^{\gamma} A_{dit}^{\alpha-1}} = L_t$

From here I numerically solve for detrended (divided by $L_t^{\frac{1}{1-\alpha-\gamma}}$) $x_{cit}, x_{dit}, A_{cit}, A_{dit}$.

Derivation of the corner solution of the Balanced growth path $z_{cit} = 0, z_{dit} = L_t$:

$$H_{C_{it}} = 0: \quad \frac{C_{it}^{-1/\sigma}}{\int_0^1 C_{it}^{\frac{\sigma-1}{\sigma}} di} \equiv \frac{C_{it}^{-1/\sigma}}{C_t^{\frac{\sigma-1}{\sigma}}} = \lambda_t$$

$$H_{x_{cit}} = 0: \quad \lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi \right] = 0$$

$$H_{x_{dit}} = 0: \quad \lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi \right] +$$
$$+ \xi_t (1-\delta) \beta A_{dit}^{1-\beta} x_{dit}^{\beta-1} = 0$$

$$H_{z_{dit}} = 0: \quad \mu_{1t} \eta_c A_{cit}^{\alpha} A_{dit}^{\gamma} < \mu_{2t} \eta_d A_{cit}^{\gamma} A_{dit}^{\alpha}$$

$$H_{A_{cit}} = (\rho - n)\mu_{1t} - \dot{\mu_{1t}}:$$
$$\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}} \right] +$$
$$+ \mu_{1t} \alpha \eta_c z_{cit} A_{cit}^{\alpha-1} A_{dit}^{\gamma} + \mu_{2t} \gamma \eta_d z_{dit} A_{cit}^{\gamma-1} A_{dit}^{\alpha} = (\rho - n)\mu_{1t} - \dot{\mu}_{1t}$$

$$H_{A_{dit}} = (\rho - n)\mu_{2t} - \dot{\mu_{2t}}:$$
$$\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}} \right] +$$
$$+ \mu_{1t} \gamma \eta_c z_{cit} A_{cit}^{\alpha} A_{dit}^{\gamma-1} + \mu_{2t} \alpha \eta_d z_{dit} A_{cit}^{\gamma} A_{dit}^{\alpha-1} + \xi_t (1-\delta)(1-\beta) A_{dit}^{-\beta} x_{dit}^{\beta} =$$
$$= (\rho - n)\mu_{2t} - \dot{\mu}_{2t}$$

$$H_{S_t} = (\rho - n)\xi_t - \dot{\xi}_t:$$
$$-\frac{\theta}{S_t} = (\rho - n)\xi_t - \dot{\xi}_t$$

Using that $z_{cit} = 0, z_{dit} = L_t$:

- $\dfrac{C_{it}^{-1/\sigma}}{C_t^{\frac{\sigma-1}{\sigma}}} = \lambda_t$
- $\mu_{1t} < \mu_{2t} \dfrac{\eta_d}{\eta_c} \left(\dfrac{A_{dit}}{A_{cit}} \right)^{\alpha-\gamma}$
- $\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} \beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi = 0$
- $\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}} \right] +$
  $+ \mu_{2t} \gamma \eta_d z_{dit} A_{cit}^{\gamma-1} A_{dit}^{\alpha} = (\rho - n)\mu_{1t} - \dot{\mu}_{1t}$
- $\lambda_t \left[\left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}-1} (1-\beta) A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} A_{dit}^{-1} x_{dit} \right] +$
  $+ \mu_{2t} \alpha \eta_d z_{dit} A_{cit}^{\gamma} A_{dit}^{\alpha-1} = (\rho - n)\mu_{2t} - \dot{\mu}_{2t}$
- $-\dfrac{\theta}{S_t \xi_t} = (\rho - n) - \dfrac{\dot{\xi}_t}{\xi_t}$

Substituting an expression for ξ_t and simplifying further (on the dirty corner, consump-

tion grows at the rate of the dirty technology g_{A_d} rather than at the BGP rate g_A, so $\rho - n - \dot{\xi}_t/\xi_t = \rho - n + g_{A_d}$):

- $\dfrac{C_{it}^{-1/\sigma}}{C_t^{\frac{\sigma-1}{\sigma}}} = \lambda_t$
- $\mu_{1t} < \mu_{2t}\dfrac{\eta_d}{\eta_c}\left(\dfrac{A_{dit}}{A_{cit}}\right)^{\alpha-\gamma}$
- $\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}\beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} - \psi = 0$
- $\mu_{2t} = \dfrac{\lambda_t\left[\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}(1-\beta)A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1}A_{dit}^{-1}x_{dit}\right]}{\rho - n - \alpha g_{A_d} - \frac{\dot{\mu}_{2t}}{\mu_{2t}}}$
- $\mu_{1t} = \dfrac{\lambda_t\left[\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}(1-\beta)\right]}{\rho - n - \frac{\dot{\mu}_{1t}}{\mu_{1t}}}\cdot$
  $\cdot\left(A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1}x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}} + \gamma g_{A_d}\dfrac{A_{dit}}{A_{cit}}\dfrac{A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1}A_{dit}^{-1}x_{dit}}{\rho - n - \alpha g_{A_d} - \frac{\dot{\mu}_{2t}}{\mu_{2t}}}\right)$
- $\dfrac{\theta((1-\beta)g_{A_d} + \beta g_{x_d})\left(1 + \left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}\left(\frac{x_{cit}}{x_{dit}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}}\left(1 - \beta\left(1 + \left(\frac{x_{cit}}{x_{dit}}\right)^{-1}\right)\right)\right)}{\left(1 - \left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}\left(\frac{x_{cit}}{x_{dit}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}-1}\right)} =$

  $= \rho - n + g_{A_d}$

  Simplifying further:

- $\dfrac{\left(A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1}x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}} + \gamma g_{A_d}\frac{A_{dit}}{A_{cit}}\frac{A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1}A_{dit}^{-1}x_{dit}}{\rho-n-\alpha g_{A_d}-\frac{\dot{\mu}_{2t}}{\mu_{2t}}}\right)}{\rho - n - \frac{\dot{\mu}_{1t}}{\mu_{1t}}} <$

  $< \dfrac{\left[A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1}A_{dit}^{-1}x_{dit}\right]}{\rho - n - \alpha g_{A_d} - \frac{\dot{\mu}_{2t}}{\mu_{2t}}}\dfrac{\eta_d}{\eta_c}\left(\dfrac{A_{dit}}{A_{cit}}\right)^{\alpha-\gamma}$
- $\dfrac{\theta((1-\beta)g_{A_d} + \beta g_{x_d})\left(1 + \left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}\left(\frac{x_{cit}}{x_{dit}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}}\left(1 - \beta\left(1 + \left(\frac{x_{cit}}{x_{dit}}\right)^{-1}\right)\right)\right)}{\left(1 - \left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}\left(\frac{x_{cit}}{x_{dit}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}-1}\right)} =$

  $= \rho - n + g_{A_d}$

Let's consider the last equation to figure out the relation between $\left(\frac{A_{dit}}{A_{cit}}\right)$ and $\left(\frac{x_{dit}}{x_{cit}}\right)$. For the dirty corner we have that $\left(\frac{A_{dit}}{A_{cit}}\right) \to \infty$, and we can present $\left(\frac{A_{dit}}{A_{cit}}\right)$ as $\left(\frac{A_{di0} \cdot e^{g_{A_d} t}}{A_{ci0}}\right)$. Let's assume that $\left(\frac{x_{dit}}{x_{cit}}\right) \to \infty$, $g_{A_d} = g_{x_d}$. We can present $\left(\frac{x_{dit}}{x_{cit}}\right)$ as $\left(\frac{x_{di0} \cdot e^{g_{x_d} t}}{x_{ci0}}\right)$.

- $\lim\limits_{t \to \infty} \frac{\left(1 + \left(\frac{A_{ci0}}{A_{di0} \cdot e^{g_{A_d} t}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} \left(\frac{x_{ci0}}{x_{di0} e^{g_{A_d} t}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}} \left(1 - \beta - \beta \left(\frac{x_{ci0}}{x_{di0} e^{g_{A_d} t}}\right)^{-1}\right)\right)^{\theta((1-\beta)g_{A_d} + \beta g_{x_d})}}{\left(1 - \left(\frac{A_{ci0}}{A_{di0} \cdot e^{g_{A_d} t}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} \left(\frac{x_{ci0}}{x_{di0} e^{g_{A_d} t}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}-1}\right)} \propto$

  $\propto \lim\limits_{t \to \infty} \frac{1 + C_1 \cdot e^{g_{A_d} t[-(1-\beta)\frac{\varepsilon-1}{\varepsilon} - \beta\frac{\varepsilon-1}{\varepsilon}]} \left(1 - \beta - \beta C_3 e^{g_{A_d} t}\right)}{1 - C_2 \cdot e^{g_{A_d} t[-(1-\beta)\frac{\varepsilon-1}{\varepsilon} - \beta\frac{\varepsilon-1}{\varepsilon} + 1]}} =$

  $= \lim\limits_{t \to \infty} \frac{1 + C_1 \left((1-\beta) \cdot e^{g_{A_d} t[-\frac{\varepsilon-1}{\varepsilon}]} - \beta C_3 e^{g_{A_d} t[\frac{1}{\varepsilon}]}\right)}{1 - C_2 \cdot e^{g_{A_d} t[\frac{1}{\varepsilon}]}} =$

  $= \frac{C_1 C_3 \beta}{C_2} = const = \rho - n + g_{A_d}$

  $\Rightarrow$ *no contradiction*

Now let's assume that $\left(\frac{x_{dit}}{x_{cit}}\right) \to 0$, $g_{A_d} = g_{x_c}$. We can present $\left(\frac{x_{dit}}{x_{cit}}\right)$ as $\left(\frac{x_{di0}}{x_{ci0} \cdot e^{g_{x_c} t}}\right)$.

- $\lim\limits_{t \to \infty} \frac{\left(1 + \left(\frac{A_{ci0}}{A_{di0} \cdot e^{g_{A_d} t}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} \left(\frac{x_{ci0} \cdot e^{g_{x_c} t}}{x_{di0}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}} \left(1 - \beta - \beta \left(\frac{x_{ci0} \cdot e^{g_{x_c} t}}{x_{di0}}\right)^{-1}\right)\right)^{\theta((1-\beta)g_{A_d} + \beta g_{x_d})}}{\left(1 - \left(\frac{A_{ci0}}{A_{di0} \cdot e^{g_{A_d} t}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} \left(\frac{x_{ci0} \cdot e^{g_{x_c} t}}{x_{di0}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}-1}\right)} \propto$

  $\propto \lim\limits_{t \to \infty} \frac{1 + C_1 \cdot e^{g_{A_d} t[-(1-\beta)\frac{\varepsilon-1}{\varepsilon} + \beta\frac{\varepsilon-1}{\varepsilon}]} \left(1 - \beta - \beta C_3 e^{-g_{A_d} t}\right)}{1 - C_2 \cdot e^{g_{A_d} t[-(1-\beta)\frac{\varepsilon-1}{\varepsilon} + \beta\frac{\varepsilon-1}{\varepsilon} - 1]}} =$

  $= \lim\limits_{t \to \infty} \frac{1 + C_1 \left((1-\beta) \cdot e^{g_{A_d} t\left[-\frac{\varepsilon-1}{\varepsilon} + 2\beta\frac{\varepsilon-1}{\varepsilon}\right]} - \beta C_3 e^{g_{A_d} t\left[-\frac{\varepsilon-1}{\varepsilon} + 2\beta\frac{\varepsilon-1}{\varepsilon} - 1\right]}\right)}{1 - C_2 \cdot e^{g_{A_d} t\left[-\frac{\varepsilon-1}{\varepsilon} + 2\beta\frac{\varepsilon-1}{\varepsilon} - 1\right]}} \neq \rho - n + g_{A_d}$

  $\Rightarrow$ *contradiction*

The limit is finite for $\beta < 1/2$ and infinite for $\beta > 1/2$; in both cases it differs from $\rho - n + g_{A_d}$ except for special parameter values. Thus, $\left(\frac{A_{dit}}{A_{cit}}\right) \to \infty \Rightarrow \left(\frac{x_{dit}}{x_{cit}}\right) \to \infty$.

We will further use that:

$$\left(\frac{A_{dit}}{A_{cit}}\right) = \left[\left(\frac{x_{dit}}{x_{cit}}\right)^{-\frac{\beta\frac{\varepsilon-1}{\varepsilon}}{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}} \left(1-\beta+\left(\frac{\rho-n+g_{A_d}}{\theta((1-\beta)g_{A_d}+\beta g_{x_d})}-\beta\right)\left(\frac{x_{dit}}{x_{cit}}\right)\right)^{\frac{1}{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}}\right] \cdot$$

$$\cdot \left(\frac{\rho-n+g_{A_d}}{\theta((1-\beta)g_{A_d}+\beta g_{x_d})}-1\right)^{-\frac{1}{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}}$$

The inequality becomes:

$$\bullet \quad \frac{1}{\rho-n-\alpha g_{A_d}-\frac{\dot{\mu}_{2t}}{\mu_{2t}}}\frac{\eta_d}{\eta_c}\left(\frac{A_{dit}}{A_{cit}}\right)^{\alpha-\gamma-1}\left(\frac{x_{dit}}{x_{cit}}\right) > \frac{\left(1+\gamma g_{A_d}\frac{\left(\frac{x_{dit}}{x_{cit}}\right)}{\rho-n-\alpha g_{A_d}-\frac{\dot{\mu}_{2t}}{\mu_{2t}}}\right)}{\rho-n-\frac{\dot{\mu}_{1t}}{\mu_{1t}}}$$

$$\Rightarrow \quad \frac{\frac{\eta_d}{\eta_c}\left[\left(\frac{x_{dit}}{x_{cit}}\right)^{-\frac{[\alpha-\gamma-1]\beta\frac{\varepsilon-1}{\varepsilon}}{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}+1}\left(1-\beta+\left(\frac{\rho-n+g_{A_d}}{\theta((1-\beta)g_{A_d}+\beta g_{x_d})}-\beta\right)\left(\frac{x_{dit}}{x_{cit}}\right)\right)^{\frac{[\alpha-\gamma-1]}{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}}\right]}{\left(\rho-n-\alpha g_{A_d}-\frac{\dot{\mu}_{2t}}{\mu_{2t}}\right)\left(\frac{\rho-n+g_{A_d}}{\theta((1-\beta)g_{A_d}+\beta g_{x_d})}-1\right)^{\frac{[\alpha-\gamma-1]}{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}}} >$$

$$> \frac{\left(1+\gamma g_{A_d}\frac{\left(\frac{x_{dit}}{x_{cit}}\right)}{\rho-n-\alpha g_{A_d}-\frac{\dot{\mu}_{2t}}{\mu_{2t}}}\right)}{\rho-n-\frac{\dot{\mu}_{1t}}{\mu_{1t}}}$$

For inequality to hold:

$$\left[\frac{[\alpha-\gamma-1]\left(-\beta\frac{\varepsilon-1}{\varepsilon}+1\right)}{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}+1>0, \quad \gamma\leq 0\right]$$

$$\left[\varepsilon > \frac{1-\beta(\alpha-\gamma)}{(1-\beta)(\alpha-\gamma)} = 1+\frac{1-(\alpha-\gamma)}{(1-\beta)(\alpha-\gamma)}, \quad \varepsilon>1, \quad \gamma\leq 0 \quad \right]$$

The branch $\varepsilon < 1$ is excluded: the step $\frac{A_{dit}}{A_{cit}} \to \infty \Rightarrow \frac{x_{dit}}{x_{cit}} \to \infty$ above uses $\frac{\varepsilon-1}{\varepsilon} > 0$, while with $\varepsilon < 1$ machines move toward the scarce clean input. As in Appendix A.3.1, with $\gamma < 0$ the term multiplied by γ on the right side goes to $-\infty$, so the inequality holds for any $\varepsilon > 1$.

A.4 Detailed solution of the model: decentralized equilibrium

Agents:

$$\max_{c_{it}} \int_0^\infty e^{-(\rho-n)t}\left[log(c_t) - \theta log(S_t)\right] dt, \qquad c_t = \left(\int_0^1 c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}$$
$$s.t.\ \dot{a}_t = ra_t - L_t \int_0^1 p_{it}c_{it}di + w_t L_t + T_t, \quad a_t = \int_0^1 V_{it}di$$

$\int_0^1 p_{it}c_{it}di = P_t c_t$

Firms:

$$\max_{x,z,p} V_{i0} = \int_0^\infty e^{-r_t \overbrace{\tau_{cost,t}}^{\text{cost of capital}}} \pi_{it} dt =$$
$$= \int_0^\infty e^{-r_t \tau_{cost,t} t}[\overbrace{\tau_{rev,t}}^{\text{revenue subsidy}} p_{it}y_{it} - \psi(x_{cit} + \overbrace{\tau_{br,t}}^{\text{carbon tax}} x_{dit}) - w_t(z_{cit} \overbrace{\tau_{cl,t}}^{\text{clean research tax}} + z_{dit})]dt$$
$$s.t.\ \ y_{it} = Y_t\left(\frac{p_{it}}{P_t}\right)^{-\sigma}, \quad y_{it} = \left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}$$
$$\dot{A}_{cit} = \eta_c z_{cit} A_{cit}^{\alpha} A_{dit}^{\gamma}, \quad \dot{A}_{dit} = \eta_d z_{dit} A_{dit}^{\alpha} A_{cit}^{\gamma}$$

We can set up an analogous problem, where consumers consume a final good comprised of different varieties, and consider the problem of the firm producing a variety.

Agent's problem becomes:

$$H = log(c_t) - \theta log(S_t) + \lambda_t[ra_t - L_t p_t c_t + w_t L_t + T_t]$$
FOCs:
$$\frac{1}{c_t} = L_t p_t \lambda_t$$
$$\lambda_t r_t = (\rho - n)\lambda_t - \dot{\lambda}_t$$
$$\Rightarrow \frac{\dot{C}_t}{C_t} = \frac{\dot{c}_t}{c_t} + n = r_t - \rho + n = g_C, \quad C_t = L_t c_t$$

Final good production:

$$\max_{y_{it}} \Pi_t = P_t Y_t - \int_0^1 p_{it} y_{it} di = P_t \left(\int_0^1 y_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}} - \int_0^1 p_{it} y_{it} di$$

FOC:

$$y_{it} = Y_t \left(\frac{p_{it}}{P_t} \right)^{-\sigma}$$

Individual (variety) firm's production:

$$\max_{x,z,p} \int_0^\infty e^{-r_t \tau_{cost,t} t} [\tau_{rev,t} p_{it} y_{it} - \psi(x_{cit} + \tau_{br,t} x_{dit}) - w_t(z_{cit} \tau_{cl,t} + z_{dit})] dt$$

$$s.t. \quad y_{it} = Y_t \left(\frac{p_{it}}{P_t} \right)^{-\sigma}, \quad y_{it} = \left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\dot{A}_{cit} = \eta_c z_{cit} A_{cit}^{\alpha} A_{dit}^{\gamma}, \quad \dot{A}_{dit} = \eta_d z_{dit} A_{dit}^{\alpha} A_{cit}^{\gamma}$$

We can normalize the price of the final good to 1.

Hamiltonian:

$$\begin{aligned} H &= \tau_{rev,t} p_{it} y_{it} - \psi(x_{cit} + \tau_{br,t} x_{dit}) - w_t(\tau_{cl,t} z_{cit} + z_{dit}) + \\ &+ \mu_{1t}[\eta_c z_{cit} A_{cit}^{\alpha} A_{dit}^{\gamma}] + \mu_{2t}[\eta_d z_{dit} A_{dit}^{\alpha} A_{cit}^{\gamma}] = \\ &= \tau_{rev,t} Y_t^{\frac{1}{\sigma}} \left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}\frac{\sigma-1}{\sigma}} - \\ &- \psi(x_{cit} + \tau_{br,t} x_{dit}) - w_t(\tau_{cl,t} z_{cit} + z_{dit}) + \\ &+ \mu_{1t}[\eta_c z_{cit} A_{cit}^{\alpha} A_{dit}^{\gamma}] + \mu_{2t}[\eta_d z_{dit} A_{dit}^{\alpha} A_{cit}^{\gamma}] \end{aligned}$$

FOCs for inner solution for the Balanced Growth Path:

- x_{cit} : $\tau_{rev,t} Y_t^{\frac{1}{\sigma}} \left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}\frac{\sigma-1}{\sigma}-1} \cdot$
  $\cdot \frac{\sigma-1}{\sigma} \beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} = \psi$
- x_{dit} : $\tau_{rev,t} Y_t^{\frac{1}{\sigma}} \left((A_{cit}^{(1-\beta)} x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)} x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}\frac{\sigma-1}{\sigma}-1} \cdot$
  $\cdot \frac{\sigma-1}{\sigma} \beta A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}} x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} = \psi \tau_{br,t}$
- z_{cit} : $w_t \tau_{cl,t} = \mu_{1t} \eta_c A_{cit}^{\alpha} A_{dit}^{\gamma}$
- z_{dit} : $w_t = \mu_{2t} \eta_d A_{dit}^{\alpha} A_{cit}^{\gamma} \Rightarrow \mu_{1t} \frac{\eta_c}{\eta_d} \left(\frac{A_{cit}}{A_{dit}} \right)^{\alpha-\gamma} = \tau_{cl,t} \mu_{2t}$

- $A_{cit}:$ $\frac{\tau_{rev,t}Y_t^{\frac{1}{\sigma}}}{\mu_{1t}}\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}+(A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}\frac{\sigma-1}{\sigma}-1}\cdot$
$\cdot\frac{\sigma-1}{\sigma}(1-\beta)A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1}x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}}+$
$+\eta_c z_{cit}\alpha A_{cit}^{\alpha-1}A_{dit}^{\gamma}+\frac{\mu_{2t}}{\mu_{1t}}\gamma\eta_d z_{dit}A_{cit}^{\gamma-1}A_{dit}^{\alpha}=\tau_{cost,t}r_t-\frac{\dot{\mu}_{1t}}{\mu_{1t}}$
- $A_{dit}:$ $\frac{\tau_{rev,t}Y_t^{\frac{1}{\sigma}}}{\mu_{2t}}\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}+(A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}\frac{\sigma-1}{\sigma}-1}\cdot$
$\cdot\frac{\sigma-1}{\sigma}(1-\beta)A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1}x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}}+$
$+\frac{\mu_{1t}}{\mu_{2t}}\eta_c z_{cit}\gamma A_{cit}^{\alpha}A_{dit}^{\gamma-1}+\eta_d z_{dit}\alpha A_{cit}^{\gamma}A_{dit}^{\alpha-1}=\tau_{cost,t}r_t-\frac{\dot{\mu}_{2t}}{\mu_{2t}}$

Simplifying further and adding a market clearing condition for researchers:

- $\tau_{rev,t}Y_t^{\frac{1}{\sigma}}\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}+(A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}\frac{\sigma-1}{\sigma}-1}\frac{\sigma-1}{\sigma}\beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1}=\psi$
- $\left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}\left(\frac{x_{cit}}{x_{dit}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}-1}=\frac{1}{\tau_{br,t}}$
$\tau_{rev,t}Y_t^{\frac{1}{\sigma}}\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}+(A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}\frac{\sigma-1}{\sigma}-1}$
- $\frac{\frac{\sigma-1}{\sigma}(1-\beta)A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1}x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}}}{\tau_{cost,t}r_t-\frac{\dot{\mu}_{1t}}{\mu_{1t}}-\eta_c A_{cit}^{\alpha-1}A_{dit}^{\gamma}\left[z_{cit}\alpha+\frac{\gamma}{\tau_{cl,t}}z_{dit}\right]}\frac{\eta_c}{\eta_d}\left(\frac{A_{cit}}{A_{dit}}\right)^{\alpha-\gamma}=$
$\tau_{rev,t}Y_t^{\frac{1}{\sigma}}\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}+(A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}\frac{\sigma-1}{\sigma}-1}$
$=\tau_{cl,t}\frac{\frac{\sigma-1}{\sigma}(1-\beta)A_{dit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1}x_{dit}^{\beta\frac{\varepsilon-1}{\varepsilon}}}{\tau_{cost,t}r_t-\frac{\dot{\mu}_{2t}}{\mu_{2t}}-\eta_d A_{cit}^{\gamma}A_{dit}^{\alpha-1}\left[\tau_{cl,t}\gamma z_{cit}+\alpha z_{dit}\right]}$
- $g_{A_c}\frac{1}{\eta_c A_{cit}^{\alpha-1}A_{dit}^{\gamma}}+g_{A_d}\frac{1}{\eta_d A_{cit}^{\gamma}A_{dit}^{\alpha-1}}=L_t$

Using that growth rates are: $\frac{\dot{\mu}_{1t}}{\mu_{1t}} = \frac{\dot{\mu}_{2t}}{\mu_{2t}} = 0,\ r_t = g_C + \rho - n = g_A + \rho - n$

- $\tau_{rev,t}\left((A_{cit}^{(1-\beta)}x_{cit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{dit}^{(1-\beta)}x_{dit}^{\beta})^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}\frac{\sigma-1}{\sigma}\beta A_{cit}^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}x_{cit}^{\beta\frac{\varepsilon-1}{\varepsilon}-1} = \psi$
- $\left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}}\left(\frac{x_{cit}}{x_{dit}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}-1} = \frac{1}{\tau_{br,t}}$
- $\left(\frac{A_{cit}}{A_{dit}}\right)^{(1-\beta)\frac{\varepsilon-1}{\varepsilon}-1+\alpha-\gamma}\left(\frac{x_{cit}}{x_{dit}}\right)^{\beta\frac{\varepsilon-1}{\varepsilon}}\frac{\eta_c}{\eta_d} =$

  $= \tau_{cl,t}\dfrac{\tau_{cost,t}[\rho-n+g_A] - \alpha g_A - \frac{\eta_c}{\eta_d}\frac{\gamma}{\tau_{cl,t}}g_A\left(\frac{A_{cit}}{A_{dit}}\right)^{\alpha-1-\gamma}}{\tau_{cost,t}[\rho-n+g_A] - \alpha g_A - \frac{\eta_d}{\eta_c}\tau_{cl,t}\gamma g_A\left(\frac{A_{cit}}{A_{dit}}\right)^{-\alpha+1+\gamma}}$
- $g_{A_c}\frac{1}{\eta_c A_{cit}^{\alpha-1}A_{dit}^{\gamma}} + g_{A_d}\frac{1}{\eta_d A_{cit}^{\gamma}A_{dit}^{\alpha-1}} = L_t$

From here I numerically solve for detrended (divided by $L_t^{\frac{1}{1-\alpha-\gamma}}$) $x_{cit}, x_{dit}, A_{cit}, A_{dit}$. With $\gamma = 0$ the numerator and the denominator of the right side of the third equation coincide, so $\tau_{cost,t}$ does not affect A_{cit}/A_{dit} on the balanced growth path.

A.5 Additional results for decentralized equilibrium

The figures for the unstable equilibrium use $\gamma = -1/15$ (so that $\varepsilon = 4 > \varepsilon^* = 3.25$ and $\gamma < 0$); all other parameters are as in the basic calibration, with $\eta_c/\eta_d = 1.1$ (left panels) and $1/1.1$ (right panels). For the unstable equilibrium, the figures show the cutoff ratios.

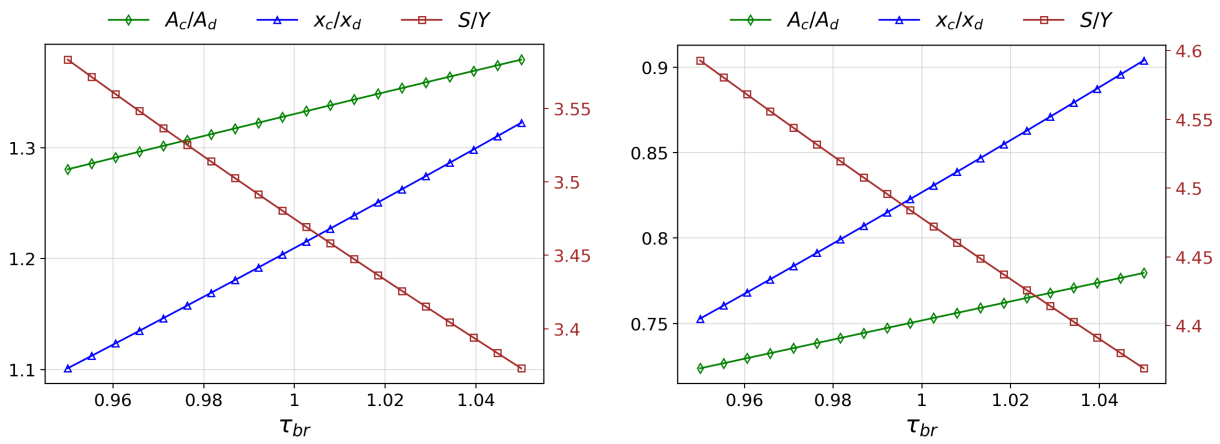
- *Increase in dirty input tax ($\tau_{br,t} > 1$) reduces emissions.*

  With an increase in the dirty input tax for the **unstable equilibrium**, the cutoff ratio of productivities becomes lower, meaning that the economy is more likely to converge to the clean corner.

- *Increase in clean research tax ($\tau_{cl,t} > 1$) increases emissions.*

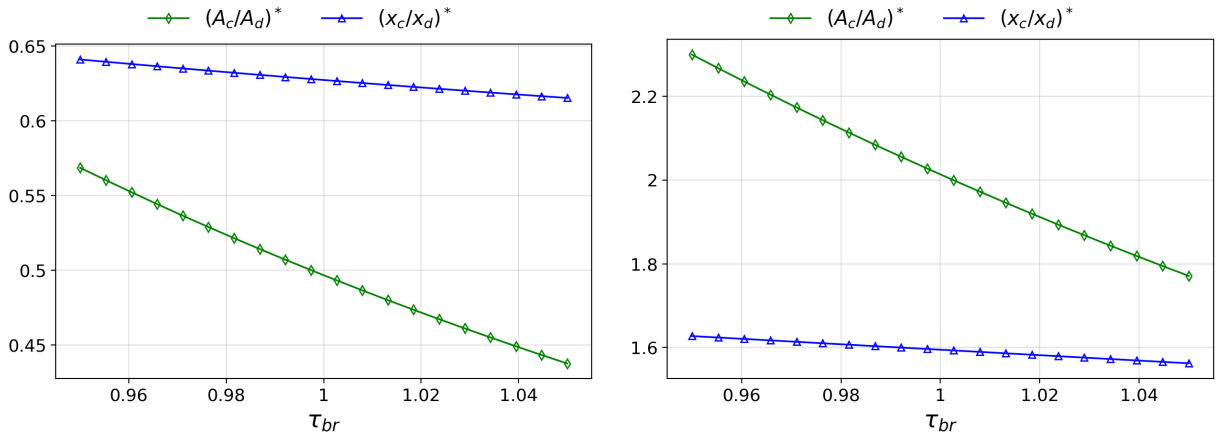
  With an increase of the clean *research tax* for the **unstable equilibrium** the cutoff ratio of productivities becomes higher, meaning that the economy is less likely to converge to the clean corner. The opposite holds for an increase in the clean research subsidy.

- *Increase in the cost of capital $\tau_{cost,t} > 1$ amplifies the existing asymmetry in clean versus dirty research productivity.*



$\eta_c > \eta_d$ $\eta_c < \eta_d$

Figure 8. Change in tax on the use of dirty machines $\tau_{br,t}$ in a **stable equilibrium**



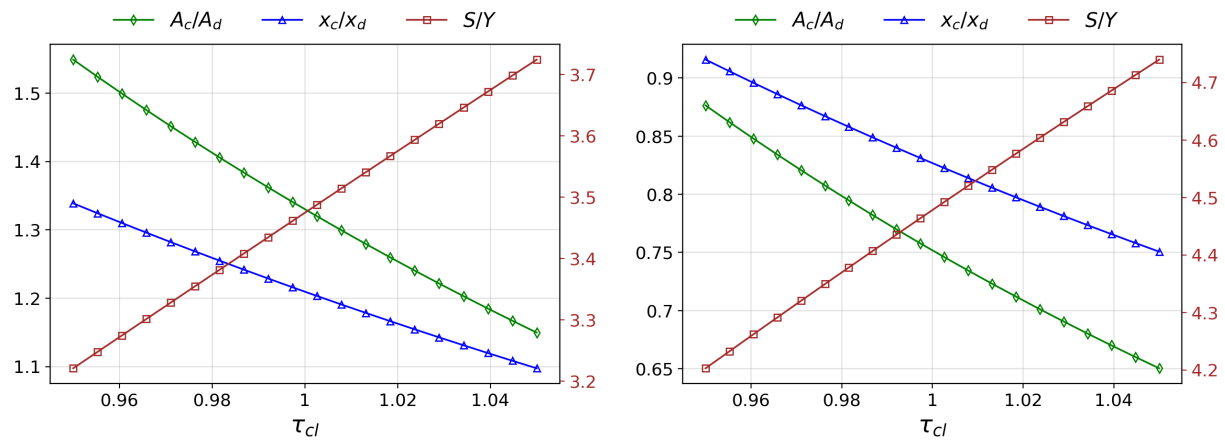
$\eta_c > \eta_d$ $\eta_c < \eta_d$

Figure 9. Change in tax on the use of dirty machines $\tau_{br,t}$ in an **unstable equilibrium**

With an increase in the cost of capital with an advancement in clean innovation productivity ($\eta_c > \eta_d$) for the **unstable equilibrium**, the cutoff ratio of productivities becomes lower, meaning that the economy is more likely to converge to the clean corner. And vice versa for the case of $\eta_c < \eta_d$.

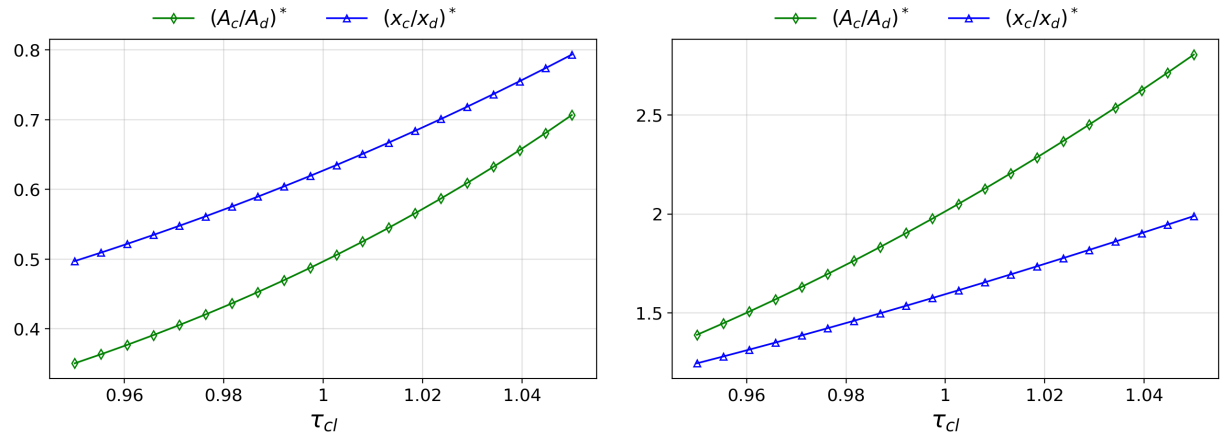
A.6 Existence, uniqueness and stability of the balanced growth path

In this section, I prove the properties of the balanced growth path (BGP) that are used in Section 2. Unless stated otherwise, I consider the social planner who does not internalize



$\eta_c > \eta_d$ $\eta_c < \eta_d$

Figure 10. Change in clean research tax $\tau_{cl,t}$ in a **stable equilibrium**



$\eta_c > \eta_d$ $\eta_c < \eta_d$

Figure 11. Change in clean research tax $\tau_{cl,t}$ in an **unstable equilibrium**

emissions ($\theta = 0$). By Lemma 2 below, the decentralized equilibrium without taxes has the same research allocation, so all the results also apply to it.

A.6.1 Reduced form of the planner's problem

Let's first solve the static choice of machines. After this choice, output net of machines is a CES function of the two technologies, which makes the dynamic problem much simpler.

Lemma 1 (Net output)**.** *For given technologies,* $\max_{x_c,x_d}[y-\psi(x_c+x_d)] = \kappa\left(A_c^{\varphi}+A_d^{\varphi}\right)^{1/\varphi}$*, where* $\kappa > 0$ *is a constant,* $\varphi = 1-1/\sigma_A$ *and* $\sigma_A = (1-\beta)\varepsilon+\beta$*. Thus, the two technologies are substitutes in net output with elasticity* σ_A*, and* ε^* *is the value of* ε *at which* $\sigma_A(\alpha-\gamma) = 1$*.*

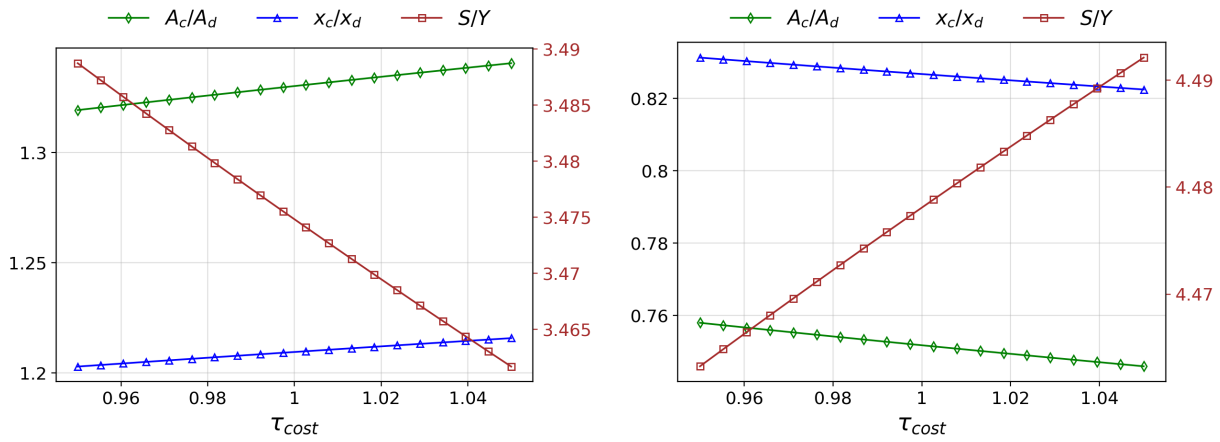


Figure 12. Increase in the cost of capital $\tau_{cost,t}$ in a **stable equilibrium**

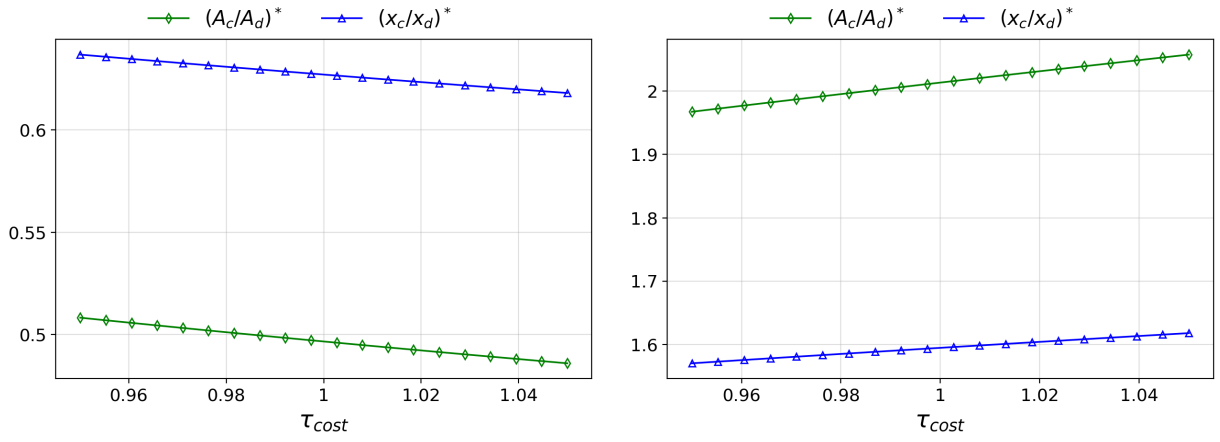


Figure 13. Increase in the cost of capital $\tau_{cost,t}$ in an **unstable equilibrium**

Proof. Denote $q_j = A_j^{1-\beta} x_j^{\beta}$ and $b = \frac{\varepsilon-1}{\varepsilon}$, so that $y = (q_c^b + q_d^b)^{1/b}$. The first-order condition for x_j is $\beta y^{1-b} q_j^b / x_j = \psi$. Combining it with the definition of q_j gives

$$q_j = A_j^{\frac{1-\beta}{1-b\beta}} \left(\frac{\beta y^{1-b}}{\psi} \right)^{\frac{\beta}{1-b\beta}}, \qquad j \in \{c, d\}.$$

Substituting q_c and q_d into $y^b = q_c^b + q_d^b$, we get $y \propto (A_c^{\varphi} + A_d^{\varphi})^{1/\varphi}$ with $\varphi = \frac{b(1-\beta)}{1-b\beta} = 1 - \frac{1}{(1-\beta)\varepsilon+\beta}$. Summing the two first-order conditions, $\psi(x_c + x_d) = \beta y^{1-b}(q_c^b + q_d^b) = \beta y$, so net output equals $(1-\beta)y$. Finally, $\sigma_A(\alpha - \gamma) = 1$ means $(1-\beta)\varepsilon + \beta = \frac{1}{\alpha-\gamma}$, that is, $\varepsilon = \frac{1-\beta(\alpha-\gamma)}{(1-\beta)(\alpha-\gamma)} = \varepsilon^*$. □

Now let's detrend the technologies, $a_{jt} = A_{jt}/L_t^{1/(1-\alpha-\gamma)}$, and denote $\ell_j = \log a_j$, the share of researchers in clean innovation $s = z_c/L$, and $g = g_A = n/(1-\alpha-\gamma)$. By Lemma 1, flow utility is the log of net output per capita, so, up to terms that do not depend on the allocation, the planner's problem becomes autonomous:

$$\max_{s_t\in[0,1]} \int_0^\infty e^{-(\rho-n)t} u(\ell_t)\, dt, \qquad u(\ell) = \tfrac{1}{\varphi}\log\left(e^{\varphi\ell_c} + e^{\varphi\ell_d}\right),$$

$$\dot\ell_c = sP_c - g, \qquad \dot\ell_d = (1-s)P_d - g, \qquad P_c = \eta_c a_c^{\alpha-1}a_d^{\gamma}, \quad P_d = \eta_d a_d^{\alpha-1}a_c^{\gamma}.$$

Here P_c and P_d are the productivities of a researcher in each sector, and the planner only chooses how to split researchers between the two sectors.

Lemma 2 (Decentralized equilibrium)**.** *(i) Without taxes, the decentralized equilibrium has the same research allocation and the same path of technologies as the planner with $\theta = 0$. (ii) With a cost-of-capital wedge τ_{cost} (any revenue subsidy, and $\tau_{br} = \tau_{cl} = 1$), the decentralized equilibrium has the same research allocation as a planner with CRRA utility with curvature τ_{cost} and discount rate $\tau_{cost}\rho$.*

Proof. (i) The firm differs from the planner in two ways: the markup and the discount rate. First, given technologies, the markup makes the firm use fewer machines, but it scales both types of machines by the same factor $\left(\frac{\sigma-1}{\sigma}\right)^{1/(1-\beta)}$, because $\partial y/\partial x_j$ is homogeneous of degree $\beta-1$ in (x_c, x_d). Thus, the marginal products of A_c and A_d are scaled by the same factor $\left(\frac{\sigma-1}{\sigma}\right)^{\beta/(1-\beta)}$, and consumption is proportional to the planner's net output. Second, the firm values technologies with costates in units of goods, μ_j, which are discounted at the interest rate: $\dot\mu_j = r_t\mu_j -$ (marginal profit from A_j), where $r_t = \rho - n + g_{C,t}$ by the Euler equation (Appendix A.4). The planner's costates in units of goods, p_j/λ_t with $\lambda_t = 1/C_t$, satisfy the same equation, because $\dot\lambda_t/\lambda_t = -g_{C,t}$. Since the marginal products are proportional and the discount rates are equal, the switching conditions $\mu_1\eta_c A_c^\alpha A_d^\gamma \gtrless \mu_2\eta_d A_d^\alpha A_c^\gamma$ of the firm and of the planner coincide along any path, so research is allocated in the same way.

(ii) With CRRA utility of curvature σ_u and discount rate ρ', the planner discounts costates in units of goods at $\rho' - \sigma_u n + \sigma_u g_{C,t}$. Setting $\sigma_u = \tau_{cost}$ and $\rho' = \tau_{cost}\rho$ gives $\tau_{cost}(\rho - n + g_{C,t}) = \tau_{cost} r_t$, which is the rate at which the firm discounts. □

A.6.2 Existence and uniqueness

Let's rewrite the BGP condition from Appendix A.3.1 in terms of one variable, $R = A_c/A_d$. The second equation of the system there gives $x_c/x_d = R^\varphi$. Substituting it into the

third equation and taking logs, the BGP ratio solves

$$f(R) \equiv \log k + e_1 \log R + \log D_2(R) - \log D_1(R) = 0, \qquad D_1, D_2 > 0,$$

where $k = \eta_c/\eta_d$, $e_1 = \alpha - \gamma - 1/\sigma_A$, $T = \rho - n + (1-\alpha)g$, $m(R) = kR^{-(1+\gamma-\alpha)}$ is the relative productivity of clean researchers P_c/P_d, and $D_1 = T - \gamma g\, m$ and $D_2 = T - \gamma g/m$ are the denominators of μ_{1t} and μ_{2t} in Appendix A.3.1. Also, $T - \gamma g = \rho - n + (1 - \alpha - \gamma)g = \rho$.

Proposition A1 (Existence and uniqueness)**.** *(i) A BGP exists if $\gamma > 0$; if $\gamma < 0$ and $\varepsilon \neq 1$; or if $\gamma = 0$ and $e_1 \neq 0$. (ii) The BGP is unique if $\gamma > 0$ and $e_1 < 2\gamma(1 + \gamma - \alpha)\frac{g_A}{\rho}$, or if $\gamma < 0$ and $e_1 > 2\gamma(1 + \gamma - \alpha)\frac{g_A}{\rho}$; with $\gamma = 0$ it is always unique. (iii) Otherwise there can be three BGPs.*

Proof. Denote $\chi = \log R$; m is decreasing in R.

Existence. If $\gamma > 0$, both denominators are positive only for $\gamma g/T < m < T/(\gamma g)$, and this interval is not empty because $T > \gamma g$. As R falls, m rises to $T/(\gamma g)$, $D_1 \to 0$ and $f \to +\infty$; as R rises, m falls to $\gamma g/T$, $D_2 \to 0$ and $f \to -\infty$. Since f is continuous, it has a zero. If $\gamma < 0$, both denominators are always positive, and for large $|\chi|$ one of them grows like m or $1/m$ while the other goes to T, so $f(\chi) \approx (e_1 + 1 + \gamma - \alpha)\chi = \varphi\chi$ at both ends. Thus, f changes sign when $\varphi \neq 0$, that is, when $\varepsilon \neq 1$. If $\gamma = 0$, $f = \log k + e_1\chi$ is linear and has a zero when $e_1 \neq 0$.

Uniqueness. Differentiating f,

$$f'(\chi) = e_1 - (1 + \gamma - \alpha)\, h(m), \qquad h(m) = \gamma g \left[\frac{m}{D_1} + \frac{1}{m\, D_2}\right].$$

The function h takes the same value at m and at $1/m$. For $\gamma > 0$ it is smallest at $m = 1$, and for $\gamma < 0$ it is largest at $m = 1$; in both cases $h(1) = \frac{2\gamma g}{T - \gamma g} = \frac{2\gamma g}{\rho}$. Thus, for $\gamma > 0$ we have $f'(\chi) \leq e_1 - 2\gamma(1 + \gamma - \alpha)g/\rho$, and for $\gamma < 0$ we have $f'(\chi) \geq e_1 - 2\gamma(1 + \gamma - \alpha)g/\rho$. Under the conditions in (ii), f is strictly monotone, so its zero is unique. □

For example, with $\alpha = 1/3$, $\gamma = -1/15$, $\eta_c = \eta_d$ and $\varepsilon = 2$ there are three BGPs, with ratios A_c/A_d equal to 0.001, 1 and 980 (clean research shares of 1.6%, 50% and 98.4%); with $\gamma = 1/12$ and $\varepsilon = 12$ the three ratios are 0.27, 1 and 3.67. In the basic calibration ($\alpha = 1/3$, $\gamma = 1/6$) the BGP is unique for every ε, because $e_1 < \alpha - \gamma = 1/6$ while $2\gamma(1 + \gamma - \alpha)g_A/\rho = 0.28$. In the unstable calibration ($\gamma = -1/15$, $\varepsilon = 4$) the BGP is unique as well. With $\gamma = -1/15$ the condition fails for $\varepsilon < 2.8$, and for every ε between 1 and 2.8 there are three BGPs; the outer two move away quickly as ε falls (for $\varepsilon = 1.5$ they are at $A_c/A_d \approx e^{\pm 12}$, practically corners). The bound in (ii) is the same expression as the stability

condition in part (ii) of the Proposition in Section 2. Thus, with $\eta_c = \eta_d$: if $\gamma > 0$, the BGP is unique whenever the symmetric BGP is stable; if $\gamma < 0$ and $\varepsilon > 1$, there are three or more BGPs exactly when the symmetric one is stable.

A.6.3 Local stability

At a BGP both sectors do research, so at the margin the planner is indifferent between clean and dirty researchers. The research share s enters the Hamiltonian linearly, and its coefficient, the switching function Φ, is zero along the BGP. In this case the usual second-order condition does not apply, and the necessary condition for an interior research allocation to be optimal is the generalized Legendre–Clebsch (Kelley) condition: $K \equiv \partial\ddot{\Phi}/\partial s \geq 0$. With $\gamma = 0$ this condition plays the role of concavity: it holds when flow utility, as a function of the technology ratio, has its peak at the BGP (Proposition A3).

Proposition A2 (Local stability of the symmetric BGP)**.** *Let $\eta_c = \eta_d$ and consider the symmetric BGP, $R^* = 1$. The research allocation on this BGP satisfies the generalized Legendre–Clebsch condition in strict form, $K > 0$, if and only if $f'(0) < 0$, that is,*

$$\alpha - \gamma - \frac{1}{(1-\beta)\varepsilon + \beta} < 2\gamma(1 + \gamma - \alpha)\frac{g_A}{\rho}.$$

If the inequality is reversed, $K < 0$ and staying on the BGP is not optimal. With $\gamma = 0$ this condition is $\varepsilon < \varepsilon^*$.

Proof. Step 1: Hamiltonian and costates. The current-value Hamiltonian of the reduced problem is $H = u(\ell) + p_c(sP_c - g) + p_d((1-s)P_d - g)$, where p_c and p_d are the costates of ℓ_c and ℓ_d. Denote $\pi = \partial u/\partial \ell_c = R^\varphi/(1+R^\varphi)$, the share of the clean technology in net output. Using $\partial P_c/\partial \ell_c = (\alpha-1)P_c$, $\partial P_c/\partial \ell_d = \gamma P_c$ and the symmetric expressions for P_d, the costate equations are

$$\begin{aligned}
\dot{p}_c &= (\rho - n)p_c - \pi + (1-\alpha)sP_c p_c - \gamma(1-s)P_d p_d, \\
\dot{p}_d &= (\rho - n)p_d - (1-\pi) - \gamma sP_c p_c + (1-\alpha)(1-s)P_d p_d.
\end{aligned}$$

The switching function is $\Phi = \partial H/\partial s = p_c P_c - p_d P_d$.

Step 2: the first derivative of Φ does not depend on s. Denote $\Phi_c = p_c P_c$ and $\Phi_d = p_d P_d$, so that $\Phi = \Phi_c - \Phi_d$. Using $\dot{P}_c/P_c = (\alpha-1)\dot{\ell}_c + \gamma\dot{\ell}_d$ and $\dot{P}_d/P_d = \gamma\dot{\ell}_c + (\alpha-1)\dot{\ell}_d$, and

$\rho - n + (1-\alpha-\gamma)g = \rho$, we get

$$\dot{\Phi}_c = \rho\,\Phi_c - \pi P_c + \gamma(1-s)(P_d\Phi_c - P_c\Phi_d),$$
$$\dot{\Phi}_d = \rho\,\Phi_d - (1-\pi)P_d - \gamma s(P_d\Phi_c - P_c\Phi_d).$$

Subtracting, the terms with s cancel:

$$\dot{\Phi} = \rho\,\Phi - \pi P_c + (1-\pi)P_d + \gamma(P_d\Phi_c - P_c\Phi_d).$$

Step 3: the second derivative. Along the BGP $\Phi = \dot{\Phi} = 0$, so $\Phi_c = \Phi_d \equiv \Psi$. In $\ddot{\Phi}$, the control s enters only through the time derivatives of π, P_c, P_d, Φ_c and Φ_d:

$$\frac{\partial\dot{\pi}}{\partial s} = \varphi\pi(1-\pi)(P_c+P_d), \qquad \frac{\partial\dot{\Phi}_c}{\partial s} = \frac{\partial\dot{\Phi}_d}{\partial s} = -\gamma\Psi(P_d - P_c),$$
$$\frac{\partial\dot{P}_c}{\partial s} = -P_c\left[(1-\alpha)P_c + \gamma P_d\right], \qquad \frac{\partial\dot{P}_d}{\partial s} = P_d\left[(1-\alpha)P_d + \gamma P_c\right].$$

Collecting the terms, we get

$$\begin{aligned} K = &-\varphi\pi(1-\pi)(P_c+P_d)^2 + (1-\alpha)\left(\pi P_c^2 + (1-\pi)P_d^2\right) + \gamma P_c P_d \\ &+ \gamma\Psi\left[4\gamma P_c P_d + (1-\alpha-\gamma)(P_c^2+P_d^2)\right]. \end{aligned}$$

Step 4: the symmetric BGP. With $\eta_c = \eta_d$ and $R = 1$ we have $\pi = 1/2$, $s = 1/2$ and $P_c = P_d = P = 2g$ (because $sP_c = g$). On the BGP $\dot{p}_c = 0$ and $p_c = p_d$, so the first costate equation gives $p_c\left[\rho - n + (1-\alpha)g - \gamma g\right] = \frac{1}{2}$, that is, $p_c = \frac{1}{2\rho}$ and $\Psi = p_c P = \frac{g}{\rho}$. Substituting into K:

$$K = P^2\left[(1-\alpha+\gamma)\left(1 + \frac{2\gamma g}{\rho}\right) - \varphi\right].$$

Since $\varphi = 1 - 1/\sigma_A$, $K > 0$ if and only if $\alpha - \gamma - 1/\sigma_A < 2\gamma(1+\gamma-\alpha)g/\rho$. At $R = 1$ we have $m = 1$, so from Appendix A.6.2 $f'(0) = e_1 - 2\gamma(1+\gamma-\alpha)g/\rho$, and thus $K = -P^2 f'(0)$. With $\gamma = 0$ the condition is $e_1 = \alpha - 1/\sigma_A < 0$, which is $\varepsilon < \varepsilon^*$ by Lemma 1. □

At an asymmetric BGP, substituting the BGP conditions into K in the same way gives

$$K = P_d^2\left[\pi(1-\pi)(1+m)^2\left(-f'(\chi^*)\right) - \frac{\gamma^2 g\,\pi(1+m)(m-1)^2}{D_1}\right],$$

where the second term is never positive because $D_1 > 0$. Thus, a BGP where f is increasing never satisfies the condition, and with $\gamma = 0$ the condition is $\varepsilon < \varepsilon^*$ for any η_c/η_d. With $\gamma \neq 0$ the condition at an asymmetric BGP is slightly stronger than $f' < 0$; in the cases I

computed, the two differ only near parameter values at which two BGPs merge, where f' is close to zero.

A.6.4 Global dynamics when $\gamma = 0$

Proposition A3 (Global dynamics when $\gamma = 0$)**.** *Let $\gamma = 0$, and let the initial technologies satisfy $\eta_d a_{c0}^{1-\alpha} + \eta_c a_{d0}^{1-\alpha} = \eta_c \eta_d / g$, which holds on the BGP; that is, only the ratio $R_0 = A_{c0}/A_{d0}$ starts off the BGP. (a) If $\varepsilon < \varepsilon^*$, the optimal path is the most rapid approach to the BGP (Spence and Starrett, 1975): all researchers work in the sector that is below its BGP level until the BGP is reached, and from then on research is split as on the BGP. (b) If $\varepsilon > \varepsilon^*$ and $R_0 > R^*$, the path with all researchers in clean innovation (toward the clean corner) gives higher utility than any path that stays above the BGP; symmetrically, if $R_0 < R^*$, the path toward the dirty corner gives higher utility than any path that stays below the BGP.*

Proof. With $\gamma = 0$ the two technologies can be described by one variable. Denote $w_j = a_j^{1-\alpha}$. Then

$$\dot{w}_c = (1-\alpha)(s\eta_c - g\, w_c), \qquad \dot{w}_d = (1-\alpha)((1-s)\eta_d - g\, w_d).$$

Step 1: one state variable. The weighted sum $Q = \eta_d w_c + \eta_c w_d$ does not depend on the research allocation: $\dot{Q} = (1-\alpha)(\eta_c \eta_d - gQ)$. Under the stated initial condition Q stays at $\eta_c \eta_d / g$ forever, so $w_d = (\eta_c \eta_d / g - \eta_d w_c)/\eta_c$ is a linear function of w_c, and the planner effectively chooses only the path of w_c.

Step 2: one-dimensional problem. Flow utility becomes $F(w_c) = \frac{1}{\varphi} \log(w_c^\nu + w_d^\nu)$ with $\nu = \varphi/(1-\alpha)$, and the problem is

$$\max_{s_t \in [0,1]} \int_0^\infty e^{-(\rho - n)t} F(w_{ct})\, dt \qquad \text{s.t.} \qquad \dot{w}_c = (1-\alpha)(s\eta_c - g\, w_c).$$

Any interior w_c can be kept constant with $s = g w_c / \eta_c$; the fastest way up is $s = 1$ and the fastest way down is $s = 0$.

Step 3: the shape of F. Since w_d is linear in w_c, $w_c^\nu + w_d^\nu$ is concave in w_c when $0 < \nu < 1$ and convex when $\nu > 1$ or $\nu < 0$. The transformation $\frac{1}{\varphi}\log(\cdot)$ is increasing when $\varphi > 0$ and decreasing when $\varphi < 0$. Thus, F has a single peak when $\nu < 1$ and is U-shaped when $\nu > 1$. With $\gamma = 0$, $\nu < 1 \Leftrightarrow \varphi < 1 - \alpha \Leftrightarrow e_1 < 0 \Leftrightarrow \varepsilon < \varepsilon^*$. The peak (or the bottom) of F is the BGP: the condition $F'(w_c) = 0$ is the BGP condition $f = 0$ with $\gamma = 0$.

Step 4: comparison of paths. (a) Let $\varepsilon < \varepsilon^*$, so F has a single peak at the BGP level w_c^*, and let $w_{c0} < w_c^*$ (the other case is symmetric). Let $\hat{w}_t$ be the path with $s = 1$ until it reaches w_c^*, and with w_c kept at w_c^* afterwards. Any feasible path grows at most as fast, $\dot{w}_c \le (1-\alpha)(\eta_c - g\, w_c)$, so $w_{ct} \le \hat{w}_t \le w_c^*$ until $\hat{w}$ reaches w_c^*. Since F is increasing below

w_c^*, $F(w_{ct}) \leq F(\hat{w}_t)$ at every date, and after $\hat{w}$ reaches w_c^* it gets the maximum of F. Thus, the most rapid approach gives higher utility at every date, so it is optimal for any discount rate. (b) Let $\varepsilon > \varepsilon^*$, so F is lowest at the BGP and increasing above it. For $w_{c0} > w_c^*$ the path with $s = 1$ rises fastest, so by the same comparison it gives higher utility at every date than any path that stays above w_c^*. □

For example, in case D of Table 4 ($\gamma = 0$, $\varepsilon = 2.5 < \varepsilon^* = 4$, $\eta_c = \eta_d$), if A_{c0}/A_{d0} starts at $e^{-0.3} \approx 0.74$ of its BGP value, all researchers work in clean innovation for 9.5 years, and then research is split equally between the two sectors. From $e^{-1} \approx 0.37$ of the BGP value, the first phase lasts 28 years.

A.6.5 Global numerical solution

The first-order conditions in Appendix A.3 and Propositions A1–A2 are only necessary. The maximized Hamiltonian is the larger of two functions of the technologies, one for each use of researchers, so it has a kink at the BGP and is not concave there; thus, Arrow's sufficiency condition does not hold. Therefore, I also solve the planner's problem globally, by value-function iteration on a grid of log technology levels (281 points in each dimension, 41 research shares, 4-year steps). Then I simulate the optimal paths starting from $\log R_0 - \log R^* \in \{\pm 0.3, \pm 1\}$, with $\log A_c$ and $\log A_d$ moved by equal amounts in opposite directions from their BGP levels, and record where they are after 600 years. Table 4 reports the results ($\eta_c = \eta_d = 0.1$, $\alpha = 1/3$, other parameters as in the basic calibration).

Table 4. Global solution of the planner's problem: long-run outcome of optimal paths

Case	γ	ε	Prediction (Prop. A2)	Outcome of optimal paths
A	1/6	4	stable	converge to the BGP
B	1/6	12	stable	converge to the BGP
C	0	6	unstable ($\varepsilon > \varepsilon^* = 4$)	corners, in the direction of R_0
D	0	2.5	stable	converge to the BGP
E	−1/15	4	unstable	corners, in the direction of R_0
F	−1/15	3	unstable ($\varepsilon < \varepsilon^* = 3.25$)	corners, in the direction of R_0
G	−1/15	2	stable (middle of 3 BGPs)	stay near the symmetric BGP
H	1/12	12	unstable at symmetric BGP	specialized interior BGPs, $\log R = \pm 1.3$

All eight cases agree with Proposition A2. Case F shows that with $\gamma < 0$ the economy goes to the corners already below ε^* (for $\gamma = -1/15$ the threshold is $\varepsilon = 2.8$). Case H shows that with $\gamma > 0$ research never concentrates in one sector, but the economy can settle on an interior BGP where one technology dominates.

A.6.6 The planner with emissions

With emissions, the planner has a third state variable, the stock of emissions. I solve this problem globally with the same method. The states are $\log(A_c/A_d)$, $\log a_d$ and $\log(S/a_d)$, and the controls are the research share and the emission wedge ω, the shadow cost of emissions in goods per unit of dirty output, which sets the use of dirty machines: $\partial y/\partial x_d = \psi + \omega\, \partial q_d/\partial x_d$ (grid $141 \times 113 \times 25$). With $\theta = 0$ this solver reproduces cases A, C and D of Table 4. Table 5 reports the results with $\theta = 0.03$.

Table 5. Global solution of the planner's problem with emissions ($\theta = 0.03$)

Case	γ	ε	BGP $\log R^*$	Outcome of optimal paths
E1	1/6	4	0.084	converge to the BGP; emission wedge $\omega \approx 0.03$
E2	0	2.5	0.193	converge to the BGP within 600 years
E3	$-1/15$	4	-0.24	corners; a start at $\log R_0 = -0.54$ goes to the clean corner

In case E1 the optimal emission wedge, $\omega = 0.03$, is close to $\psi(\tau_{br} - 1)/(\partial q_d/\partial x_d) = 0.035$, the wedge implied by the dirty input tax $\tau_{br} = 1.03$ that replicates the planner's BGP in Section 3. In case E3, the starting ratio that separates the two corners is below the BGP ratio: emissions make the dirty corner less attractive, so the planner chooses the clean corner even from $\log R_0 = -0.54 < \log R^* = -0.24$. Thus, the initial ratio that separates the two corners is the ratio at which the planner is indifferent between them, and it equals R^* only in symmetric cases.

Consider also the first-order conditions along the clean corner with emissions. Along this path, the planner reduces dirty output faster, the stock of emissions stops growing, and the social value of dirty knowledge goes to zero. With $\gamma = 0$ and $\varepsilon = 2.5 < \varepsilon^* = 4$, the switching condition for the clean corner fails at every date without emissions, but with $\theta = 0.03$ it holds from about year 540 (and from about year 1,550 when $\varepsilon = 1.5$). With $\gamma = 1/6$ it fails at every date, as without emissions. Thus, with emissions the clean corner satisfies the first-order conditions in more cases. However, these conditions are only necessary: in case E2 ($\gamma = 0$, $\varepsilon = 2.5$) the global solution still converges to the mixed BGP, even from $\log R_0 = 1.19$, one unit above it.

A.6.7 Decentralized equilibrium with taxes

Proposition A4 (Decentralized equilibrium with taxes)**.** *With constant taxes: (i) the BGP ratio solves* $f_{DE}(R) \equiv f(R; \hat{k}, T') + \frac{\beta b}{1-\beta b} \log \tau_{br} = 0$, *with* $b = \frac{\varepsilon - 1}{\varepsilon}$, *where* $f(\cdot; \hat{k}, T')$ *is the function* f *from Appendix A.6.2 with* $\hat{k} = k/\tau_{cl}$ *and* $T' = \tau_{cost}(\rho - n + g) - \alpha g$ *in place of* k *and* T, *and Proposition A1 applies with these substitutions and with* ρ *in its bounds replaced*

by $T' - \gamma g$; (ii) the corner conditions are the same as in Section 2: taxes change only the constants in the switching inequality, not the exponents of R; (iii) with $\tau_{br} = \tau_{cl} = 1$, the research allocation is that of the CRRA planner of Lemma 2(ii).

Proof. (i) Let's start from the last system in Appendix A.4. The second equation gives $x_c/x_d = (\tau_{br} R^{(1-\beta)b})^{1/(1-\beta b)}$. Substituting it into the third equation, the left side becomes $k\,\tau_{br}^{\beta b/(1-\beta b)} R^{e_1}$, and the right side is $\tau_{cl} D_1'/D_2'$ with $D_1' = T' - \gamma g\, m/\tau_{cl}$ and $D_2' = T' - \gamma g\, \tau_{cl}/m$. Dividing both sides by τ_{cl} and taking logs gives $f_{DE}(R) = 0$, where m/τ_{cl} is the relative productivity computed with $\hat{k}$. (ii) Along a corner path, the private machine ratio has the same exponent of R as in the planner's problem, and the denominators of the costates converge to constants, so the derivation in Appendix A.3.1 goes through with different constants. (iii) This is Lemma 2(ii). □

The cost of capital therefore changes the set of BGPs through T'. With $\eta_c = \eta_d$, two specialized BGPs appear when the slope of f_{DE} at the symmetric BGP turns positive. With $\gamma = 1/12$ this happens at $\varepsilon = 10$ when $\tau_{cost} = 1$, at $\varepsilon = 9.5$ when $\tau_{cost} = 1.05$, and at $\varepsilon = 7.6$ when $\tau_{cost} = 1.5$. Proposition A2 is proven for the planner with log utility, so for the CRRA planner of Lemma 2(ii) I check stability with the global solution. For $\varepsilon = 8.5$, the optimal paths converge to the mixed BGP when $\tau_{cost} = 1$, but move to the specialized BGPs at $\log R = \pm 1.6$ when $\tau_{cost} = 1.5$, which are roots of f_{DE}. With $\tau_{br} \neq 1$ or $\tau_{cl} \neq 1$, the equilibrium conditions are not the first-order conditions of any planner's problem: τ_{cl} enters the two cross-spillover terms with the factors $1/\tau_{cl}$ and τ_{cl}, and τ_{br} distorts the static choice of machines, so consumption is no longer proportional to the planner's net output. Thus, I do not characterize their transition dynamics; parts (i) and (ii) still hold.